Rigged configurations for generalized Kac–Moody algebras

Ben Salisbury

Central Michigan University

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Joint work with Travis Scrimshaw (The University of Queensland)

Let

- A be an even, integral, symmetrizable Borcherds–Cartan matrix;
- $\mathfrak{g}$ be a generalized Kac–Moody algebra associated to A; and
- $B(\infty)$ be the crystal basis of $U_q^-(\mathfrak{g})$.

Main results

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Theorem (S–Scrimshaw, 2015, 2018, 2021)

- *There is a combinatorial model for $B(\infty)$ given by rigged configurations.*
- *The $*$-crystal structure on $B(\infty)$ can be computed using the combinatorics of rigged configurations.*

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Theorem (S–Scrimshaw, 2018, 2021)

There exists a list of conditions which characterize $B(\infty)$ completely and makes the role of the $*$-crystal operators explicit.

Borcherds–Cartan matrices

Let I be a countable set. A *Borcherds–Cartan matrix* $A = (A_{ab})_{a,b\in I}$ is a real matrix such that

1. $A_{aa} = 2$ or $A_{aa} \leq 0$ for $a \in I$,
2. $A_{ab} \leq 0$ if $a \neq b$,
3. $A_{ab} \in \mathbf{Z}$ if $A_{aa} = 2$, and
4. $A_{ab} = 0 \iff A_{ba} = 0$.

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- An index $a \in I$ is called *real* if $A_{aa} = 2$ and is called *imaginary* if $A_{aa} \leq 0$.
- The subset of I of all real (resp. imaginary) indices is denoted I^{re} (resp. I^{im}).
- We will always assume that $A_{ab} \in \mathbf{Z}$, $A_{aa} \in \{2\} \cup 2\mathbf{Z}_{<0}$, and that A is symmetrizable.

Example

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Example (Borcherds, 1992)

Let $I = \{(i,t) : i \in \mathbf{Z}_{\geq -1},\ 1 \leq t \leq c(i)\}$, where $c(i)$ is the i-th coefficient of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots = \sum_{i \geq -1} c(i)q^i.$$

Define $A = (A_{(i,t),(j,s)})$, where each entry is defined by

$$A_{(i,t),(j,s)} = -(i+j).$$

Definition (Jeong–Kang–Kashiwara–Shin, 2007)

An *abstract $U_q(\mathfrak{g})$-crystal* is a nonempty set B together with maps

$$\underbrace{e_a,\ f_a \colon B \longrightarrow B \sqcup \{\mathbf{0}\}}_{\text{(Kashiwara operators)}}, \quad \varepsilon_a,\ \varphi_a \colon B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \underbrace{\mathrm{wt}\colon B \longrightarrow P}_{\text{(weight map)}},$$

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for $a \in I$, subject to the conditions for abstract crystals associated to Kac–Moody algebras, with the following changes:

1. For any $a \in I$ and $v \in B$ such that $e_a v \neq \mathbf{0}$,
   a. $\varepsilon_a(e_a v) = \varepsilon_a(v) - 1$ and $\varphi_a(e_a v) = \varphi_a(v) + 1$ if $a \in I^{\mathrm{re}}$,
   b. $\varepsilon_a(e_a v) = \varepsilon_a(v)$ and $\varphi_a(e_a v) = \varphi_a(v) + A_{aa}$ if $a \in I^{\mathrm{im}}$.
2. For any $a \in I$ and $v \in B$ such that $f_a v \neq \mathbf{0}$,
   a. $\varepsilon_a(f_a v) = \varepsilon_a(v) + 1$ and $\varphi_a(f_a v) = \varphi_a(v) - 1$ if $a \in I^{\mathrm{re}}$,
   b. $\varepsilon_a(f_a v) = \varepsilon_a(v)$ and $\varphi_a(f_a v) = \varphi_a(v) - A_{aa}$ if $a \in I^{\mathrm{im}}$.

The crystal $B(\infty)$ (Jeong–Kang–Kashiwara, 2005)

The negative half $U_q^-(\mathfrak{g})$ has an associated abstract $U_q(\mathfrak{g})$-crystal, denoted

$$B(\infty) = \{f_{a_1} \cdots f_{a_r} \mathbf{1} : r \geq 0,\ a_1, \dots, a_r \in I\}.$$

Here, $\mathbf{1} \in B(\infty)$ is the unique element such that $\mathrm{wt}(\mathbf{1}) = 0$.

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Moreover, for all $v \in B(\infty)$ and $a, a_1, \dots, a_r \in I$, we have

$$\varepsilon_a(v) = \begin{cases} \max\{k \geq 0 : e_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_a(v) = \varepsilon_a(v) + \langle h_a, \mathrm{wt}(v)\rangle,$$

$$\mathrm{wt}(f_{a_1} \cdots f_{a_r}\mathbf{1}) = -\alpha_{a_1} - \cdots - \alpha_{a_r}.$$

$*$-involution on $B(\infty)$

There is a $\mathbf{Q}(q)$-antiautomorphism $*\colon U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ defined by

$$E_a^* = E_a, \qquad F_a^* = F_a, \qquad q^* = q, \qquad (q^h)^* = q^{-h}.$$

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Theorem (Lusztig, 1990; Kashiwara, 1993; Lamprou, 2012)

Let $B(\infty)^*$ be the image of $B(\infty)$ under $*$. Then $B(\infty)^* = B(\infty)$.

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Theorem (Lusztig, 1990; Kashiwara, 1993; Lamprou, 2012)

Let $B(\infty)^*$ be the image of $B(\infty)$ under $*$. Then $B(\infty)^* = B(\infty)$.

This induces a new crystal structure on $B(\infty)$ given by

$$e_a^* = * \circ e_a \circ *, \qquad f_a^* = * \circ f_a \circ *, \qquad \varepsilon_a^* = \varepsilon_a \circ *, \qquad \varphi_a^* = \varphi_a \circ *,$$

and weight function wt being the usual weight function on $B(\infty)$.

Rigged configurations

A *rigged configuration* is a pair (ν, J) consisting of

- a multipartition $\nu = (\nu^{(a)} : a \in I)$ and
- a collection $J = (J_i^{(a)} : a \in I,\ i \in \mathbf{Z}_{\geq 0})$ of multisets of integers

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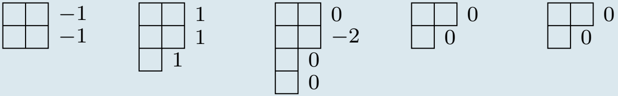

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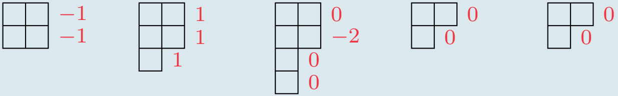

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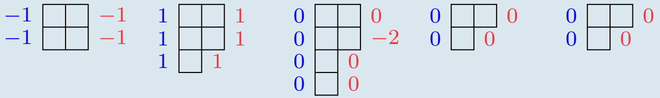

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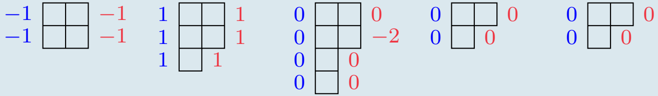

- The numbers on the right of the partition correspond to J and are called *riggings* or *labels*.
- The numbers of the left are the *vacancy numbers* $p_i^{(a)}$.
- The numbers $p_i^{(a)} - x$ are the *coriggings* or *colabels*.

Computing e_i

Definition (Schilling, 2006; S–Scrimshaw, 2021)

Fix some $a \in I$. Let x be the smallest rigging in $(\nu, J)^{(a)}$.

- Suppose $a \in I^{\mathrm{re}}$. If $x = 0$, then $e_a(\nu, J) = \mathbf{0}$. Otherwise, let r be a row in $(\nu, J)^{(a)}$ of minimal length ℓ with rigging x.
- Suppose $a \in I^{\mathrm{im}}$. If $\nu^{(a)} = \emptyset$ or $x \neq -A_{aa}/2$, then $e_a(\nu, J) = \mathbf{0}$. Otherwise let r be the row with rigging $-A_{aa}/2$.

If $e_a(\nu, J) \neq \mathbf{0}$, then $e_a(\nu, J)$ is the rigged configuration that removes a box from row r, sets the new rigging of r to be $x + A_{aa}/2$, and changes all other riggings such that the coriggings remain fixed.

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Let $I = \{1, 2\}$, $A = \left(\begin{smallmatrix} 2 & -3 \\ -1 & -4 \end{smallmatrix}\right)$, and
$$(\nu, J) = \begin{array}{rll} 1 & \boxed{}\boxed{}\boxed{} & 1 \\ 5 & \boxed{} & -1 \end{array} \qquad \begin{array}{rll} 14 & \boxed{} & 12 \\ 14 & \boxed{} & 7 \\ 14 & \boxed{} & 3 \end{array} .$$

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Then

$$e_1(\nu, J) = \begin{array}{rll} 3 & \boxed{}\boxed{}\boxed{} & 3 \end{array} \qquad \begin{array}{rll} 13 & \boxed{} & 11 \\ 13 & \boxed{} & 6 \\ 13 & \boxed{} & 2 \end{array}$$

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Computing f_i

Definition (Schilling, 2006; S–Scrimshaw, 2021)

Fix some $a \in I$. Let x be the smallest rigging in $(\nu, J)^{(a)}$. Let r be a row in $(\nu, J)^{(a)}$ of maximal length ℓ with rigging x. Then $f_a(\nu, J)$ is the rigged configuration that adds a box to row r, sets the new rigging of r to be $x - A_{aa}/2$, and changes all other riggings such that the coriggings remain fixed.

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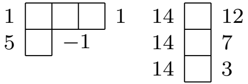
Let $I = \{1, 2\}$, $A = \left(\begin{smallmatrix} 2 & -3 \\ -1 & -4 \end{smallmatrix}\right)$, and $(\nu, J) =$  .

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and

$$f_2(\nu, J) = \begin{array}{rll} 4 & \square\square\square & 4 \\ 8 & \square & 2 \end{array} \qquad \begin{array}{rll} 18 & \square & 16 \\ 18 & \square & 11 \\ 18 & \square & 7 \\ 18 & \square & 2 \end{array}.$$

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Fix some $a \in I$. Let x be the smallest corigging in $(\nu, J)^{(a)}$.

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$$e_1^*(\nu, J) = \mathbf{0} \qquad \text{and} \qquad e_2^*(\nu, J) = \begin{array}{r|c|c|c|l} -2 & & & & 1 \\ 2 & & \multicolumn{3}{l}{-1} \end{array} \quad \begin{array}{r|c|l} 10 & & 7 \\ 10 & & 3 \end{array} .$$

Computing f_i^*

Definition (S–Scrimshaw, 2018, 2021)

Fix some $a \in I$. Let x be the smallest corigging in $(\nu, J)^{(a)}$. Let r be a row in $(\nu, J)^{(a)}$ of maximal length ℓ with corigging x. Then $f_a^*(\nu, J)$ is the rigged configuration that adds a box to row r, sets the rigging of r so that the corigging is $x - A_{aa}/2$, and keeps all other riggings fixed.

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Let $I = \{1, 2\}$, $A = \begin{pmatrix} 2 & -3 \\ -1 & -4 \end{pmatrix}$, and $(\nu, J) = \begin{array}{r|c|c|c|l} \cline{2-4} 1 & & & & 1 \\ \cline{2-4} 5 & & \multicolumn{3}{l}{-1} \\ \cline{2-2} \end{array} \quad \begin{array}{r|c|l} \cline{2-2} 14 & & 12 \\ \cline{2-2} 14 & & 7 \\ \cline{2-2} 14 & & 3 \\ \cline{2-2} \end{array}$.

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$$f_1^*(\nu, J) = \begin{array}{rll} -1 & \square\square\square\square & 0 \\ 5 & \square & -1 \end{array} \quad \begin{array}{rll} 14 & \square & 12 \\ 14 & \square & 7 \\ 14 & \square & 3 \end{array}$$

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and

$$f_2^*(\nu, J) = \begin{array}{rll} 4 & \square\square\square & 1 \\ 8 & \square & -1 \end{array} \quad \begin{array}{rll} 18 & \square & 16 \\ 18 & \square & 12 \\ 18 & \square & 7 \\ 18 & \square & 3 \end{array}.$$

Define $(\nu_\emptyset, J_\emptyset)$ by

- $\nu_\emptyset^{(a)} = 0$ for all $a \in I$,
- $J_i^{(a)} = 0$ for all $(a,i) \in I \times \mathbf{Z}_{>0}$.

Crystal model for $B(\infty)$

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Definition

Define $\mathrm{RC}(\infty)$ to be the graph generated by $(\nu_\emptyset, J_\emptyset)$, e_a, and f_a, for $a \in I$.

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Definition

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The remainder of the crystal structure is given by

$$\varepsilon_a(\nu, J) = \begin{cases} \max\{k \in \mathbf{Z} : e_a^k(\nu, J) \neq 0\} & \text{if } a \in I^{\mathrm{re}} \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases}$$
$$\varphi_a(\nu, J) = \langle h_a, \mathrm{wt}(\nu, J)\rangle + \varepsilon_a(\nu, J),$$
$$\mathrm{wt}(\nu, J) = -\sum_{a \in I} |\nu^{(a)}| \alpha_a.$$

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Define the remaining crystal structure by

$$\begin{aligned}
\varepsilon_a^*(\nu, J) &= \begin{cases} \max\{k \in \mathbf{Z} : (e_a^*)^k(\nu, J) \neq 0\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases} \\
\varphi_a^*(\nu, J) &= \langle h_a, \mathrm{wt}(\nu, J)\rangle + \varepsilon_a^*(\nu, J), \\
\mathrm{wt}(\nu, J) &= -\sum_{a\in I} |\nu^{(a)}|\alpha_a.
\end{aligned}$$

Theorem (S–Scrimshaw, 2018, 2021)

As $U_q(\mathfrak{g})$-crystals, $\mathrm{RC}(\infty) \cong \mathrm{RC}(\infty)^* \cong B(\infty)$ *and*

$$e_a^* = * \circ e_a \circ *, \qquad f_a^* = * \circ f_a \circ *.$$

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Corollary

The $*$-involution on $\mathrm{RC}(\infty)$ is given by replacing every rigging x of a row of length i in $(\nu, J)^{(a)}$ by the corresponding corigging $p_i^{(a)} - x$ for all $a \in I$ and $i > 0$.

Irreducible highest weight crystals

Associated to each irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ in $\mathcal{O}^{\mathrm{int}}$ is an abstract $U_q(\mathfrak{g})$-crystal, denoted

$$B(\lambda) = \{f_{a_1} \cdots f_{a_r} u_\lambda : r \geq 0,\ a_1, \dots, a_r \in I\} \setminus \{\mathbf{0}\}.$$

Here, $u_\lambda \in B(\lambda)$ is the unique element such that $\mathrm{wt}(u_\lambda) = \lambda$.

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Here, $u_\lambda \in B(\lambda)$ is the unique element such that $\mathrm{wt}(u_\lambda) = \lambda$.

In this case, for all $a, a_1, \cdots, a_r \in I$ and $v \in B(\lambda)$, we have

$$\varepsilon_a(v) = \begin{cases} \max\{k \geq 0 : e_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_a(v) = \begin{cases} \max\{k \geq 0 : f_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{im}}, \end{cases}$$

$$\mathrm{wt}(f_{a_1} \cdots f_{a_r} u_\lambda) = \lambda - \alpha_{a_1} - \cdots - \alpha_{a_r}.$$

Theorem (Jeong–Kang–Kashiwara–Shin, 2007)

Let $\lambda \in P^+$ and let T_λ and C be certain "elementary crystals." Then $B(\lambda)$ is isomorphic to the connected component of $B(\infty) \otimes T_\lambda \otimes C$ containing $\mathbf{1} \otimes t_\lambda \otimes c$.

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- Define new crystal operators $f'_a(\nu, J)$ as $f_a(\nu, J)$ unless
  - $p_i^{(a)} + \langle h_a, \lambda \rangle < x$ for some $(a,i) \in \mathcal{H}$ and $x \in J_i^{(a)}$ or
  - $\varphi_a(\nu, J) = 0$ for $a \in I^{\mathrm{im}}$,

  in which case set $f'_a(\nu, J) = 0$.
- Let $\mathrm{RC}(\lambda)$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under f'_a.

Irreducible highest weight crystals and rigged configurations

Theorem (Jeong–Kang–Kashiwara–Shin, 2007)

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  in which case set $f'_a(\nu, J) = 0$.
- Let $\mathrm{RC}(\lambda)$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under f'_a.

Theorem (Schilling, 2006; S-Scrimshaw, 2015, 2017, 2021)

Let $\lambda \in P^+$. Then $\mathrm{RC}(\lambda) \cong B(\lambda)$.

One can characterize the image of $B(\lambda)$ inside $B(\infty)$ using the $*$-involution in analogy to Kashiwara (1995).

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Corollary

Let $\lambda \in P^+$. Then we have

$$\mathrm{RC}(\lambda) \cong \left\{ (\nu, J) \otimes t_\lambda \in \mathrm{RC}(\infty) \otimes T_\lambda : \substack{\varepsilon_a^*(\nu,J) \le \langle h_a,\lambda\rangle \text{ for all } a \in I^{\mathrm{re}}, \\ e_a^*(\nu,J) = \mathbf{0} \text{ if } \langle h_a,\lambda\rangle = 0 \text{ for all } a \in I^{\mathrm{im}}} \right\}.$$

Characterizing $B(\infty)$

Theorem (Kashiwara–Saito, 1997; Jeong–Kang–Kashiwara–Shin, 2007)

Let B *be an abstract* $U_q(\mathfrak{g})$*-crystal such that*

1. $\mathrm{wt}(B) \subset -Q^+$,
2. *there exists an element* $v_0 \in B$ *such that* $\mathrm{wt}(v_0) = 0$,
3. *for any* $v \in B$ *such that* $v \neq v_0$, *there exists some* $a \in I$ *such that* $e_a v \neq 0$, *and*
4. *for all* $a \in I$, *there exists a strict embedding* $\Psi_a \colon B \longrightarrow B \otimes \mathbf{N}_{(a)}$.

Then there exists a crystal isomorphism $B \cong B(\infty)$ *such that* $v_0 \mapsto \mathbf{1}$.

Characterizing $B(\infty)$ again

Define

$$\widetilde{\varepsilon}_a(v) := \max\{k' \geq 0 : e_a^{k'} v \neq \mathbf{0}\}, \qquad \widetilde{\varphi}_a(v) := \max\{k' \geq 0 : f_a^{k'} v \neq \mathbf{0}\},$$

and similarly for $\widetilde{\varepsilon}_a^*$ and $\widetilde{\varphi}_a^*$ using e_a^* and f_a^* respectively. Additionally, define

$$\kappa_a(v) := \begin{cases} \varepsilon_a(v) + \varepsilon_a^*(v) + \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{re}}, \\ \varepsilon_a(v) + \widetilde{\varepsilon}_a^*(v) A_{aa} + \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{im}}. \end{cases}$$

Theorem (Kashiwara–Saito, 1997; Tingley–Webster, 2016; S–Scrimshaw, 2018, 2021)

Let $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ *and* $(B^\star, e_a^\star, f_a^\star, \varepsilon_a^\star, \varphi_a^\star, \mathrm{wt})$ *be connected abstract* $U_q(\mathfrak{g})$*-crystals with the same highest weight element* $v_0 \in B \cap B^\star$ *that is the unique element of weight* 0*, where the remaining crystal data is determined by setting* $\mathrm{wt}(v_0) = 0$ *and*

$$\varepsilon_a(v) = \begin{cases} \max\{k \geq 0 : e_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}. \end{cases}$$

Characterizing $B(\infty)$ again (continued)

Theorem

Assume further that, for all $a \neq b$ in I and all $v \in B$,

1. $f_a v$, $f_a^\star v \neq \mathbf{0}$;
2. $f_a^\star f_b v = f_b f_a^\star v$ *and* $\widetilde{\varepsilon}_a^\star(f_b v) = \widetilde{\varepsilon}_a^\star(v)$ *and* $\widetilde{\varepsilon}_b(f_a^\star v) = \widetilde{\varepsilon}_b(v)$;
3. $\kappa_a(v) = 0$ *implies* $f_a v = f_a^\star v$;
4. *for* $a \in I^{\mathrm{re}}$:
   - $\kappa_a(v) \geq 0$;
   - $\kappa_a(v) \geq 1$ *implies* $\varepsilon_a^\star(f_a v) = \varepsilon_a^\star(v)$ *and* $\varepsilon_a(f_a^\star v) = \varepsilon_a(v)$;
   - $\kappa_a(v) \geq 2$ *implies* $f_a f_a^\star v = f_a^\star f_a v$;
5. *for* $a \in I^{\mathrm{im}}$: $\kappa_a(v) > 0$ *implies* $\widetilde{\varepsilon}_a^\star(f_a v) = \widetilde{\varepsilon}_a^\star(v)$ *and* $f_a f_a^\star v = f_a^\star f_a v$.

Then

$$(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt}) \cong B(\infty).$$

Characterizing $B(\infty)$ again (continued... again)

Theorem

Moreover, suppose $\kappa_a(v) = 0$ *if and only if*

$$\kappa_a^\star(v) := \varepsilon_a^*(v) + \widetilde{\varepsilon}_a(v) A_{aa} + \langle h_a, \mathrm{wt}(v) \rangle = 0$$

for all $a \in I^{\mathrm{im}}$ *and* $v \in B$. *Then*

$$(B^\star, e_a^\star, f_a^\star, \varepsilon_a^\star, \varphi_a^\star, \mathrm{wt}) \cong B(\infty)$$

with $e_a^\star = e_a^*$ *and* $f_a^\star = f_a^*$.

THANK YOU!