Since $p \geq q$, there is one less $\left(\boxed{n}, \boxed{\overline{n}}\right)$, one more $\boxed{0}$, and one more unpaired $\boxed{\overline{n}}$, so

$$\begin{aligned}\Psi(f_n^{\mathcal{T}} R_j) &= 2(q-1)(\beta_{j,n}) + (\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) \\ &= (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).\end{aligned}$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 2: $p < q$, $z = 0$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + 2p(\beta_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\,\boxed{\cdots}\,\boxed{n}}_{q-1}\,\underbrace{\boxed{0}}_{1}\,\underbrace{\boxed{\overline{n}}\,\boxed{\cdots}\,\boxed{\overline{n}}}_{p}.$$

The number of $\left(\boxed{n}, \boxed{\overline{n}}\right)$ pairs is unchanged, there is one less unpaired $\boxed{n}$, and one more $\boxed{0}$, so

$$\begin{aligned}\Psi(f_n^{\mathcal{T}} R_j) &= (q-p-1)(\beta_{j,n-1}) + (\beta_{j,n}) + 2(p)(\beta_{j,n}) \\ &= (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).\end{aligned}$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n-1}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 3: $p \geq q$, $z = 1$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (2q+1)(\beta_{j,n}) + (p-q)(\gamma_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\,\boxed{\cdots}\,\boxed{n}}_{q}\,\underbrace{\boxed{0}}_{0}\,\underbrace{\boxed{\overline{n}}\,\boxed{\cdots}\,\boxed{\overline{n}}}_{p+1}.$$

There is one less $\boxed{0}$ and one more unpaired $\boxed{\overline{n}}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 4: $p < q$, $z = 1$ and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\,\boxed{\cdots}\,\boxed{n}}_{q}\,\underbrace{\boxed{0}}_{0}\,\underbrace{\boxed{\overline{n}}\,\boxed{\cdots}\,\boxed{\overline{n}}}_{p+1}.$$

There is one less $\boxed{0}$, one less unpaired $\boxed{n}$, and one more $(\boxed{n}, \boxed{\overline{n}})$ pair, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n})$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n-1}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 5: $j = n$. The bracketing sequences are

$$\mathrm{br}_n(R_n) = \,)^{2p}\,)^z\,(^z\,(^{2q} \quad \text{and} \quad S_n\big(\Psi(R_n)\big) = \,)^{c_{\beta_{n,n}}}.$$

Since there is no '(' in $S_n(\Psi(R_n))$, f_n^{Kp} will add $(\beta_{n,n})$ to $\Psi(R_n)$.

If $z = 1$, then the leftmost '(' in $\mathrm{br}_n(R_n)$ comes from the $\boxed{0}$ so $f_n^{\mathcal{T}}(R_i)$ sends the $\boxed{0}$ to $\boxed{\overline{n}}$. According to Ψ, we then have that

$$\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n).$$

If $z = 0$, then the leftmost '(' comes from an $\boxed{n}$ so $f_n^{\mathcal{T}} R_n$ sends an $\boxed{n}$ to an $\boxed{0}$. Again,

$$\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n).$$

Now, consider T to be of type C_n. Assume row j of T has p $\boxed{\overline{n}}$ boxes and q $\boxed{n}$ boxes,

$$R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}.$$

The bracketing sequences are:

$$\mathrm{br}_n(R_j) = \,)^p\,(^q \quad \text{and} \quad S_n\big(\Psi(R_j)\big) = \,)^{c_{\gamma_{j,j}}}\,(^{c_{\beta_{j,n-1}}}\,)^{c_{\gamma_{j,n}}}\,(^{c_{\gamma_{j,j}}}.$$

Case 6: $p \geq q$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = q(\gamma_{j,j}) + (p-q)(\gamma_{j,n}).$$

If $q = 0$, then f_n will act on the $\boxed{n}$ in R_n of T (see Case 9 for more details in this situation). When $q > 0$ by the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

There are two more unpaired $\boxed{\overline{n}}$ and one less $(\boxed{n}, \boxed{\overline{n}})$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}).$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\gamma_{j,j}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 7: $q > p+1$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

There is one more $(\boxed{n}, \boxed{\overline{n}})$ pair and two less unpaired $\boxed{n}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}).$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n-1}$, so

$$f_n^{\mathrm{Kp}} \Psi(R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 8: $q = p + 1$, and $j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

Since $q - p = 1$, the number of $(\boxed{n}, \boxed{\overline{n}})$ pairs is unchanged. There is one less $\boxed{n}$ and one more $\boxed{\overline{n}}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}).$$

On the other hand, in $S_n^c(\Psi(R_j))$ the leftmost '(' corresponds to $\beta_{j,n-1}$, so

$$f_n^{\mathrm{Kp}} \Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 9: $j = n$. The only positive root that can be in $\Psi(R_n)$ is $(\gamma_{n,n})$, so there is no '(' in $S_n(\Psi(R_n))$, and, by Definition 2.13, f_n^{Kp} adds a $(\gamma_{n,n})$. The leftmost '(' in $\mathrm{br}_n(T)$ comes from an $\boxed{n}$, so $f_n^{\mathcal{T}}$ sends an $\boxed{n}$ to an $\boxed{\overline{n}}$. Hence, $\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\gamma_{n,n}) = f_n^{\mathrm{Kp}} \Psi(R_n)$. ■

Proof of Theorem 3.1 It suffices to show that for all i we have $f_i^{\mathrm{Kp}} \Psi(T) = \Psi(f_i^{\mathcal{T}} T)$. By the definition of the bracketing sequences and of Ψ, we have

$$\mathrm{br}_i(T) \text{ factors as } \mathrm{br}_i(R_1)\mathrm{br}_i(R_2)\cdots\mathrm{br}_i(R_n), \text{ and}$$

$$S_i\big(\Psi(T)\big) \text{ factors as } S_i\big(\Psi(R_1)\big)\, S_i\big(\Psi(R_2)\big)\cdots S_i\big(\Psi(R_n)\big).$$

Suppose that the leftmost '(' in $\mathrm{br}_i^c(T)$ comes from row R_j. There will always be an uncanceled bracket coming from row i, so we can assume $j \le i$. By applying Lemma 3.4 or Lemma 3.5 to each R_j, the leftmost '(' in $S_i(\Psi(T))$ comes from $S_i\big(\Psi(R_j)\big)$, and also $\Psi(f_i^{\mathcal{T}} R_j) = f_i^{\mathrm{Kp}} \Psi(R_j)$. The result follows. ■

4 Stack Notation

This work extends results from [3, 12] in types A_n and D_n to types B_n and C_n. The type A_n result can be described using the multisegments from [6, 9, 14], which are a diagrammatic notation that makes the crystal structure apparent. In [12], this was extended to type D_n by introducing a *stack* notation for Kostant partitions in which the crystal structure can easily be read off. We now define a similar stack notation for types B_n and C_n.

In type B_n we associate positive roots with "stacks" by

$$\beta_{j,k} = \begin{smallmatrix} k \\ \vdots \\ j \end{smallmatrix}, \qquad \gamma_{\ell,m} = \begin{smallmatrix} m \\ \vdots \\ n-1 \\ n\ n \\ n-1 \\ \vdots \\ \ell \end{smallmatrix},$$

for $1 \le j \le k \le n$ and $1 \le \ell < m \le n$.

In type C_n we associate positive roots with "stacks" by

$$\beta_{j,k} = \begin{smallmatrix} k \\ \vdots \\ j \end{smallmatrix}, \qquad \gamma_{\ell,m} = \begin{smallmatrix} m \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ \ell \end{smallmatrix}, \qquad \gamma_{h,h} = \begin{smallmatrix} & n & \\ n-1 & & n-1 \\ & \vdots & \\ h & & h \end{smallmatrix},$$

for $1 \le j \le k < n$, $1 \le \ell < m \le n$, and $1 \le h \le n$.

Then the sequences of roots Φ_i from Definition 2.11 are exactly those positive roots where we can either add or remove an i from the top of the corresponding stack and still have either a valid stack, an empty stack, or in type C_n with $i = n$ where we have two valid stacks side by side. Once the stacks are ordered as in Definition 2.11, the bracketing sequence is created by placing a left bracket for each i that can be added to the top of a stack, and a right bracket for each i that can be removed from the top. Note that if both happen, then the root corresponding to the stack appears twice in Definition 2.11, in which case the ')' is placed over the left copy and the '(' over the right copy. If there is a leftmost uncanceled '(' the crystal operator, f_i adds an i to the top of the corresponding stack (or, in the case of $i = n$ in type C_n, may combine two stacks together and attach an n at the top). Otherwise, f_i creates a new stack consisting of just i.

Remark 4.1 Being able to add or remove an i from the top of a stack is different from being able to add or remove an α_i from the corresponding root. For instance, in type B_3, if $\beta = \alpha_1 + \alpha_2 + 2\alpha_3$, then $\beta - \alpha_1$ is a root, but there is no 1 at the top of the stack corresponding to β, so β is not in Φ_1^B. Similarly, in type C_3, although $\begin{smallmatrix} 2 \\ 3 \\ 2 \\ 1 \end{smallmatrix}$ is a stack, $\alpha_1 + 2\alpha_2 + \alpha_3$ is not in Φ_1^C, because the stack for $2\alpha_1 + 2\alpha_2 + \alpha_3$ is $\begin{smallmatrix} & 3 & \\ 2 & & 2 \\ 1 & & 1 \end{smallmatrix}$, not $\begin{smallmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{smallmatrix}$.

Example 4.2 Consider type C_3 and $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ given in stack notation by

$$\boldsymbol{\alpha} = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\ \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\ \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ 1 & & 1 \end{smallmatrix}\ \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ 1 & & 1 \end{smallmatrix}\ \begin{smallmatrix} & 3 & \\ 2 & & 2 \end{smallmatrix}\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\ 3\,.$$

The corresponding 3-signature is

$$\begin{array}{rccccccccccccc} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{2\\1} & \substack{2\\1} & \substack{2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2} & \substack{3\\2} & \substack{3\\2\,2} & 3 \\ S_3(\boldsymbol{\alpha}) = &) &) & (& (& (&) &) & (& (&) &) & (&) \\ S_3^c(\boldsymbol{\alpha}) = &) &) & (& & & & & & & & & & \end{array} .$$

Thus, the action of f_3 on $\boldsymbol{\alpha}$ adds a 3 to top of a $\substack{2\\1}$. This gives

$$f_3\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2}\ \substack{2\\3}\ 3 .$$

Example 4.3 Consider type C_3 and $\boldsymbol{\alpha}$ as in Example 2.15. In stack notation,

$$\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2}\ \substack{3\\2}\ 3 .$$

Recalculating the 3-signature using stack notation gives

$$\begin{array}{rcccccccccccccc} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{2\\1} & \substack{2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2\\1\,1} & \substack{3\\2\,2} & \substack{3\\2} & \substack{3\\2\,2} & 3 \\ S_3(\boldsymbol{\alpha}) = &) &) &) & (& (&) &) & (& (& (&) &) & (&) \\ S_3^c(\boldsymbol{\alpha}) = &) &) &) & & & & & (& & & & & & \end{array} .$$

Since the leftmost '(' comes from a $\substack{3\\2\,2\\1\,1}$, we should add a 3 to the top of this stack, which gives $\substack{3\,3\\2\,2\\1\,1}$. That is not the stack of a single root, but should be thought of as two copies of $\substack{3\\2\\1}$, which is the stack of a root. The result is

$$f_3\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2}\ \substack{3\\2}\ 3 .$$

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