A rigged configuration model for $B(\infty)$

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A B S T R A C T

We describe a combinatorial realization of the crystals $B(\infty)$ and $B(\lambda)$ using rigged configurations in all symmetrizable Kac–Moody types up to certain conditions. This includes all simply-laced types and all non-simply-laced finite and affine types.

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1. Introduction

Crystal basis theory is an elegant and fruitful subject born out of the theory of quantum groups. Defined by Kashiwara in the early 1990s [13,14], crystals provide a natural combinatorial framework to study the representations of Kac–Moody algebras (including classical Lie algebras) and their associated quantum groups. Their applications span many areas of mathematics, including representation theory, algebraic combinatorics, automorphic forms, and mathematical physics, to name a few.

The study of crystal bases has led researchers to develop different combinatorial models for crystals which yield suitable settings to studying a particular aspect of the

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representation theory of quantum groups. For example, highest weight crystals (which are combinatorial skeletons of an irreducible highest weight module over a quantum group) can be modeled using generalized Young tableaux [12,18], using the Littelmann path model [26–29], using alcove paths [24,25] or alcove walks [35], using geometric methods [4,7,19], and many others. The choice of using one model over the other usually depends on the underlying question at hand (and/or on the preference of the author). In concert with the descriptions for the highest weight crystals, there are several known realizations of the (infinite) crystal $B(\infty)$ (which is a combinatorial skeleton for the Verma module with highest weight 0), both in combinatorial and geometric settings, which have various applications. Combinatorially describing the crystal $B(\infty)$ in affine types is still a work in progress (see [21,22] for a generalization of the tableaux model to the affine setting in certain types), so another combinatorial model for $B(\infty)$ in affine types may prove useful.

Our choice of model will be that of *rigged configurations*, which arise naturally as indexing the eigenvalues and eigenvectors of a Hamiltonian of a statistical model [3,20,23]. On the other hand, these eigenvectors may also be indexed by one-dimensional lattice paths [2,8,9,30,41], which can be interpreted as highest weight vectors in a tensor product of certain crystals. In recent years, the implied connection between highest weight vectors in tensor products of Kirillov–Reshetikhin crystals and rigged configurations has been worked out [32,34,37,38,40].

As we will show, the rigged configuration model has simple combinatorial rules for describing the structure which work in all finite, affine, and all simply-laced Kac–Moody types. These combinatorial rules are only based on the nodes of the Dynkin diagram and their neighbors. This allows us to easily describe the embeddings of $B(\lambda)$ into $B(\mu)$, where $\lambda_a \leq \mu_a$ for all indices a. Moreover, we can easily describe the so-called virtualization of $B(\lambda)$ inside of a highest weight crystal of simply-laced type via a diagram folding.

The purpose of this paper is to extend the crystal structure on highest weight crystals in finite type in terms of rigged configurations [6,31,37,38,40] to other types and to a crystal model for $B(\infty)$ in terms of rigged configurations. In slightly more detail, the crystal $B(\infty)$ is a direct limit of all highest weight crystals, so by relaxing a certain admissibility condition on elements of the highest weight crystal, we may obtain a representative of an element of $B(\infty)$ given by a rigged configuration. An added perk of describing $B(\infty)$ using rigged configurations is that the description is type-independent. However our proofs are *almost* type-independent as we can do our proofs uniformly across all simply-laced finite types, but there will be some changes in the extension to non-simply-laced finite types and then, again, when extending outside of finite type.

The organization of this paper goes as follows. Section 2 gives a background on crystals and rigged configurations. In Section 3, we describe the rigged configuration model for $B(\infty)$ for simply-laced finite types. In Section 4, we extend our model for arbitrary simply-laced types. We extend our model to all finite, affine, and certain indefinite (symmetrizable) types in Section 5. We describe how highest weight crystals sit inside our $B(\infty)$ model using rigged configurations in Section 6.

Notational remark. The notation $\mathfrak{g}$ may denote different objects in different sections, but we will make this clear near the beginning of each (sub)section.

2. Background

2.1. Crystals

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with index set I, generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$, weight lattice P, root lattice Q, fundamental weights $\{\Lambda_i : i \in I\}$, simple roots $\{\alpha_i : i \in I\}$, and simple coroots $\{h_i : i \in I\}$. There is a canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \longrightarrow \mathbf{Z}$ defined by $\langle h_i, \alpha_j \rangle = A_{ij}$, where $P^\vee$ is the dual weight lattice.

An *abstract $U_q(\mathfrak{g})$-crystal* is a nonempty set B together with maps

$$\mathrm{wt} \colon B \longrightarrow P, \quad \varepsilon_i, \varphi_i \colon B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad e_i, f_i \colon B \longrightarrow B \sqcup \{0\},$$

subject to the conditions

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for all $i \in I$,
(2) if $b \in B$ satisfies $e_i b \neq 0$, then
    (a) $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$,
    (b) $\varphi_i(e_i b) = \varphi_i(b) + 1$,
    (c) $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$,
(3) if $b \in B$ satisfies $f_i b \neq 0$, then
    (a) $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$,
    (b) $\varphi_i(f_i b) = \varphi_i(b) - 1$,
    (c) $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$,
(4) $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B$ and $i \in I$,
(5) if $\varphi_i(b) = -\infty$ for $b \in B$, then $e_i b = f_i b = 0$.

The operators e_i and f_i, for $i \in I$, are referred to as the *Kashiwara raising* and *Kashiwara lowering operators*, respectively. See [10,14] for details.

Example 2.1. For a dominant integral weight λ, the crystal basis

$$B(\lambda) = \{f_{i_k} \cdots f_{i_1} u_\lambda : i_1, \ldots, i_k \in I,\ k \in \mathbf{Z}_{\geq 0}\} \setminus \{0\}$$

of an irreducible, highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ is an abstract $U_q(\mathfrak{g})$-crystal. The crystal $B(\lambda)$ is characterized by the following properties.

(1) The element $u_\lambda \in B(\lambda)$ is the unique element such that $\mathrm{wt}(u_\lambda) = \lambda$.
(2) For all $i \in I$, $e_i u_\lambda = 0$.
(3) For all $i \in I$, $f_i^{\langle h_i, \lambda\rangle + 1} u_\lambda = 0$.

Example 2.2. The crystal basis

$$B(\infty) = \{f_{i_k} \cdots f_{i_1} u_\infty : i_1, \ldots, i_k \in I, \ k \in \mathbf{Z}_{\geq 0}\}$$

of the negative half $U_q^-(\mathfrak{g})$ of the quantum group is an abstract $U_q(\mathfrak{g})$-crystal. Some important properties of $B(\infty)$ are the following.

(1) The element $u_\infty \in B(\infty)$ is the unique element such that $\mathrm{wt}(u_\infty) = 0$.
(2) For all $i \in I$, $e_i u_\infty = 0$.
(3) For any sequence $(i_1, \ldots, i_k)$ from I, $f_{i_k} \cdots f_{i_1} u_\infty \neq 0$.

An abstract $U_q(\mathfrak{g})$-crystal is said to be *upper regular* if, for all $b \in B$,

$$\varepsilon_i(b) = \max\{k \in \mathbf{Z}_{\geq 0} : e_i^k b \neq 0\}.$$

Similarly, an abstract $U_q(\mathfrak{g})$-crystal is said to be *lower regular* if, for all $b \in B$,

$$\varphi_i(b) = \max\{k \in \mathbf{Z}_{\geq 0} : f_i^k b \neq 0\}.$$

If B is both upper regular and lower regular, then we say B is *regular*. In this latter case, we may depict the entire i-string through $b \in B$ diagrammatically as

$$e_i^{\varepsilon_i(b)} b \xrightarrow{\ i\ } e_i^{\varepsilon_i(b)-1} b \xrightarrow{\ i\ } \cdots \xrightarrow{\ i\ } e_i b \xrightarrow{\ i\ } b \xrightarrow{\ i\ } f_i b \xrightarrow{\ i\ } \cdots \xrightarrow{\ i\ } f_i^{\varphi_i(b)-1} b \xrightarrow{\ i\ } f_i^{\varphi_i(b)} b.$$

Note that $B(\lambda)$ is a regular abstract $U_q(\mathfrak{g})$-crystal, but $B(\infty)$ is only upper regular.

Let B_1 and B_2 be two abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\psi : B_1 \longrightarrow B_2$ is a map $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ such that

(1) $\psi(0) = 0$;
(2) if $b \in B_1$ and $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$;
(3) for $b \in B_1$, we have $\psi(e_i b) = e_i \psi(b)$ provided $\psi(e_i b) \neq 0$ and $e_i \psi(b) \neq 0$;
(4) for $b \in B_1$, we have $\psi(f_i b) = f_i \psi(b)$ provided $\psi(f_i b) \neq 0$ and $f_i \psi(b) \neq 0$.

A morphism ψ is called *strict* if ψ commutes with e_i and f_i for all $i \in I$. Moreover, a morphism $\psi : B_1 \longrightarrow B_2$ is called an *embedding* if the induced map $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ is injective.

We say an abstract $U_q(\mathfrak{g})$-crystal is simply a $U_q(\mathfrak{g})$-*crystal* if it is crystal isomorphic to the crystal basis of a $U_q(\mathfrak{g})$-module.

Again let B_1 and B_2 be abstract $U_q(\mathfrak{g})$-crystals. The tensor product $B_2 \otimes B_1$ is defined to be the Cartesian product $B_2 \times B_1$ equipped with crystal operations defined by

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \le \varphi_i(b_1), \end{cases}$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \ge \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases}$$

$$\varepsilon_i(b_2 \otimes b_1) = \max\bigl(\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_2) \rangle\bigr),$$

$$\varphi_i(b_2 \otimes b_1) = \max\bigl(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \mathrm{wt}(b_1) \rangle\bigr),$$

$$\mathrm{wt}(b_2 \otimes b_1) = \mathrm{wt}(b_2) + \mathrm{wt}(b_1).$$

Remark 2.3. Our convention for tensor products is opposite the convention given by Kashiwara in [14].

More generally if $B_1, \dots, B_t$ are regular crystals, to compute the action of the Kashiwara operators on the tensor product $B = B_t \otimes \cdots \otimes B_2 \otimes B_1$, we use the *signature rule*. Indeed, for $i \in I$ and $b = b_t \otimes \cdots \otimes b_2 \otimes b_1$ in B, write

$$\underbrace{+ \cdots +}_{\varphi_i(b_t)} \underbrace{- \cdots -}_{\varepsilon_i(b_t)} \cdots \underbrace{+ \cdots +}_{\varphi_i(b_1)} \underbrace{- \cdots -}_{\varepsilon_i(b_1)} .$$

From the above sequence, successively delete any $(-, +)$-pair to obtain a sequence

$$i\text{-}\mathrm{sgn}(b) := \underbrace{+ \cdots +}_{\varphi_i(b)} \underbrace{- \cdots -}_{\varepsilon_i(b)} .$$

Suppose $1 \le j_-, j_+ \le t$ are such that b_{j_-} contributes the leftmost $-$ in $i\text{-}\mathrm{sgn}(b)$ and b_{j_+} contributes the rightmost $+$ in $i\text{-}\mathrm{sgn}(b)$. Then

$$e_i b = b_t \otimes \cdots \otimes b_{j_-+1} \otimes e_i b_{j_-} \otimes b_{j_--1} \otimes \cdots \otimes b_1,$$

$$f_i b = b_t \otimes \cdots \otimes b_{j_++1} \otimes f_i b_{j_+} \otimes b_{j_+-1} \otimes \cdots \otimes b_1.$$

Let $\mathscr{C}$ denote the category of abstract $U_q(\mathfrak{g})$-crystals. In [17], Kashiwara showed that direct limits exist in $\mathscr{C}$. Indeed, let $\{B_j\}_{j \in J}$ be a directed system of crystals and let $\psi_{k,j} \colon B_j \longrightarrow B_k$, $j \le k$, be a crystal morphism (with $\psi_{j,j}$ being the identity map on B_j) such that $\psi_{k,j}\psi_{j,i} = \psi_{k,i}$. Let $\vec{B} = \varinjlim B_j$ be the direct limit of this system and let $\psi_j \colon B_j \longrightarrow \vec{B}$. Then $\vec{B}$ has a crystal structure induced from the crystals $\{B_j\}_{j \in J}$. Indeed, for $\vec{b} \in \vec{B}$ and $i \in I$, define $e_i \vec{b}$ to be $\psi_j(e_i b_j)$ if there exists $b_j \in B_j$ such that $\psi_j(b_j) = \vec{b}$ and $e_i(b_j) \ne 0$. This definition does not depend on the choice of b_j. If there is no such b_j, then set $e_i \vec{b} = 0$. The definition of $f_i \vec{b}$ is similar. Moreover, the functions wt, ε_i, and φ_i on B_j extend to functions on $\vec{B}$.

2.2. Rigged configurations

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with index set I. Set $\mathcal{H} = I \times \mathbf{Z}_{>0}$. Consider a multiplicity array

$$L = \left(L_i^{(a)} \in \mathbf{Z}_{\geq 0} : (a,i) \in \mathcal{H} \right)$$

and a dominant integral weight λ of $\mathfrak{g}$. We call a sequence of partitions $\nu = \{\nu^{(a)} : a \in I\}$ an (L,λ)-*configuration* if

$$\sum_{(a,i) \in \mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i) \in \mathcal{H}} i L_i^{(a)} \Lambda_a - \lambda, \tag{2.1}$$

where $m_i^{(a)}$ is the number of parts of length i in the partition $\nu^{(a)}$. The set of all such (L,λ)-configurations is denoted $C(L,\lambda)$. To an element $\nu \in C(L,\lambda)$, define the *vacancy number* of ν to be

$$p_i^{(a)} = p_i^{(a)}(\nu) = \sum_{j \geq 0} \min(i,j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}} \frac{A_{ab}}{\gamma_b} \min(\gamma_a i, \gamma_b j) m_j^{(b)}, \tag{2.2}$$

where $\{\gamma_a : a \in I\}$ are some set of positive integers. If $\mathfrak{g}$ is of simply-laced type, we take $\gamma_a = 1$ for all $a \in I$.

Recall that a partition is a multiset of integers (typically sorted in decreasing order). A *rigged partition* is a multiset of pairs of integers (i,x) such that $i > 0$ (typically sorted under decreasing lexicographic order). Each (i,x) is called a *string*, where i is called the length or size of the string and x is the *label*, *rigging*, or *quantum number* of the string. Finally, a *rigged configuration* is a pair (ν, J) where $\nu \in C(L,\lambda)$ and $J = \left(J_i^{(a)} \right)_{(a,i) \in \mathcal{H}}$ where each $J_i^{(a)}$ the weakly decreasing sequence of riggings of strings of length i in $\nu^{(a)}$. We call a rigged configuration *valid* if every label $x \in J_i^{(a)}$ satisfies the inequality $p_i^{(a)} \geq x$ for all $(a,i) \in \mathcal{H}$. We say a rigged configuration is *highest weight* if $x \geq 0$ for all labels x. Define the *colabel* or *coquantum number* of a string (i,x) to be $p_i^{(a)} - x$. For brevity, we will often denote the ath part of (ν, J) by $(\nu, J)^{(a)}$ (as opposed to $(\nu^{(a)}, J^{(a)})$).

Example 2.4. Rigged configurations will be depicted as sequences of partitions with parts labeled on the left by the corresponding vacancy number and labeled on the right by the corresponding rigging. For example,
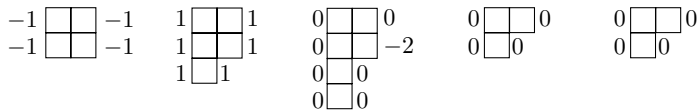
is a rigged configuration with $\mathfrak{g} = D_5$ and L is given by $L_2^{(1)} = L_1^{(2)} = L_1^{(3)} = 1$ with all other $L_i^{(a)} = 0$.

Denote by $\mathrm{RC}^*(L,\lambda)$ the set of valid highest weight rigged configurations (ν, J) such that $\nu \in C(L,\lambda)$. In [38], an abstract $U_q(\mathfrak{g})$-crystal structure was given to rigged configurations, which we recall first by defining the Kashiwara operators.

Definition 2.5. Let (ν, J) be a valid rigged configuration. Fix $a \in I$ and let x be the smallest label of $(\nu, J)^{(a)}$.

(1) If $x \geq 0$, then set $e_a(\nu, J) = 0$. Otherwise, let ℓ be the minimal length of all strings in $(\nu, J)^{(a)}$ which have label x. The rigged configuration $e_a(\nu, J)$ is obtained by replacing the string (ℓ, x) with the string $(\ell - 1, x + 1)$ and changing all other labels so that all colabels remain fixed.

(2) If $x > 0$, then add the string $(1, -1)$ to $(\nu, J)^{(a)}$. Otherwise, let ℓ be the maximal length of all strings in $(\nu, J)^{(a)}$ which have label x. Replace the string (ℓ, x) by the string $(\ell + 1, x - 1)$ and change all other labels so that all colabels remain fixed. If the result is a valid rigged configuration, then it is $f_a(\nu, J)$. Otherwise $f_a(\nu, J) = 0$.

Let $\mathrm{RC}(L, \lambda)$ denote the set generated by $\mathrm{RC}^*(L, \lambda)$ by the Kashiwara operators. For $(\nu, J) \in \mathrm{RC}(L, \lambda)$, if f_a adds a box to a string of length ℓ in $(\nu, J)^{(a)}$, then the vacancy numbers in simply-laced type are changed using the formula

$$p_i^{(b)} = \begin{cases} p_i^{(b)} & \text{if } i \leq \ell, \\ p_i^{(b)} - A_{ab} & \text{if } i > \ell. \end{cases} \tag{2.3}$$

On the other hand, if e_a removes a box from a string of length ℓ, then the vacancy numbers must be changed using

$$p_i^{(b)} = \begin{cases} p_i^{(b)} & \text{if } i < \ell, \\ p_i^{(b)} + A_{ab} & \text{if } i \geq \ell. \end{cases} \tag{2.4}$$

Let $\mathrm{RC}(L)$ be the closure under the Kashiwara operators of the set $\mathrm{RC}^*(L) = \bigcup_{\lambda \in P^+} \mathrm{RC}^*(L, \lambda)$. Lastly, the weight map $\mathrm{wt} \colon \mathrm{RC}(L) \longrightarrow P$ is defined as

$$\mathrm{wt}(\nu, J) = \sum_{(a,i) \in \mathcal{H}} i \big(L_i^{(a)} \Lambda_a - m_i^{(a)} \alpha_a \big). \tag{2.5}$$

Example 2.6. Let (ν, J) be the rigged configuration from Example 2.4. Then

and

Also we have

$$\mathrm{wt}\big((\nu, J)\big) = 2\Lambda_1 + \Lambda_2 + \Lambda_3 - 4\alpha_1 - 5\alpha_2 - 6\alpha_3 - 3\alpha_4 - 3\alpha_5$$

$$= -\Lambda_1 + \Lambda_2,$$

$$\mathrm{wt}\big(e_3(\nu, J)\big) = -\Lambda_1 + 2\Lambda_3 - \Lambda_4 - \Lambda_5 = -\Lambda_1 + \Lambda_2 + \alpha_3,$$

$$\mathrm{wt}\big(f_2(\nu, J)\big) = -\Lambda_2 + \Lambda_3 = -\Lambda_1 + \Lambda_2 - \alpha_2,$$

Theorem 2.7. *(See [38, Thm. 3.7].) Let $\mathfrak{g}$ be a simply-laced Lie algebra. For $(\nu, J) \in \mathrm{RC}^*(L, \lambda)$, let $X_{(\nu,J)}$ be the graph generated by (ν, J) and e_a, f_a for $a \in I$. Then $X_{(\nu,J)}$ is isomorphic to the crystal graph $B(\lambda)$ as $U_q(\mathfrak{g})$-crystals.*

Remark 2.8. In [38], elements of $X_{(\nu,J)}$ were called unrestricted rigged configurations and the graph $X_{(\nu,J)}$ was denoted $X_{(\bar{\nu},\bar{J})}$.

We note that our condition for highest weight rigged configurations is equivalent to the rigged configuration being highest weight in the sense of a crystal of type $\mathfrak{g}$; i.e., that the action of all e_a on a highest weight rigged configuration is 0.

In the sequel, set $\nu_\emptyset$ to be the multipartition with all parts empty; that is, set $\nu_\emptyset = (\nu^{(1)}, \ldots, \nu^{(n)})$ where $\nu_i^{(a)} = \emptyset$ for all $(a, i) \in \mathcal{H}$. Therefore the rigging $J_\emptyset$ of $\nu_\emptyset$ must be $J_i^{(a)} = \emptyset$ for all $(a, i) \in \mathcal{H}$. When discussing the highest weight crystals $X_{(\nu_\emptyset, J_\emptyset)}$, we will choose our multiplicity array L to be such that

$$\sum_{(a,i) \in \mathcal{H}} i L_i^{(a)} \Lambda_a = \lambda.$$

It is clear that there are several choices of L that may fit this condition, but this does not affect the crystal structure.

Definition 2.9. Define $\mathrm{RC}(\lambda)$ to be $X_{(\nu_\emptyset, J_\emptyset)}$ for any symmetrizable Kac–Moody algebra.

3. Rigged configuration model for $B(\infty)$ in simply-laced finite type

For this section, unless otherwise noted, let $\mathfrak{g}$ be a Lie algebra of simply-laced finite type. We wish to generate a model for $B(\infty)$ with $(\nu_\emptyset, J_\emptyset)$ as its highest weight vector. By choosing a fixed $\lambda > 0$, for any $(\nu, J) \in \mathrm{RC}(\lambda)$, there exists $k \geq 0$ such that $f_a^k(\nu, J) = 0$ by the validity condition given in Definition 2.5. Therefore, we need a modified Kashiwara operator f_a' (for $a \in I$) on rigged configurations to allow the condition $(f_a')^k(\nu, J) \neq 0$ for all $k \geq 0$. To do so, simply define f_a' by the same process given in Definition 2.5 with the validity condition omitted and choose $\lambda = 0$.

Definition 3.1. For any symmetrizable Kac–Moody algebra $\mathfrak{g}$ with index set I, define $\mathrm{RC}(\infty)$ to be the graph generated by $(\nu_\emptyset, J_\emptyset)$, e_a, and f_a', for $a \in I$, where e_a acts on elements (ν, J) in $\mathrm{RC}(\infty)$ using the same procedure as in Definition 2.5.

The remainder of the crystal structure is given by

$$\varepsilon_a(\nu, J) = \max\{k \in \mathbf{Z}_{\geq 0} : e_a^k(\nu, J) \neq 0\}, \tag{3.1a}$$

$$\varphi_a(\nu, J) = \varepsilon_a(\nu, J) + \langle h_a, \mathrm{wt}(\nu, J)\rangle, \tag{3.1b}$$

$$\mathrm{wt}(\nu, J) = -\sum_{(a,i)\in\mathcal{H}} im_i^{(a)}\alpha_a = -\sum_{a\in I} |\nu^{(a)}|\alpha_a. \tag{3.1c}$$

It is worth noting that, in this case, the definition of the vacancy numbers reduces to

$$p_i^{(a)}(\nu) = p_i^{(a)} = -\sum_{(b,j)\in\mathcal{H}} A_{ab}\min(i,j)m_j^{(b)}. \tag{3.2}$$

Example 3.2. Let $\mathfrak{g}$ be of type A_5 and (ν, J) be the rigged configuration

$$(\nu, J) = -1\ \boxed{}\ -1 \quad -2\ \boxed{}\ -1 \qquad 0\ \boxed{}\ 1 \qquad 0\ \boxed{}\ -1 \quad -3\ \boxed{}\ -1$$

Then $\mathrm{wt}(\nu, J) = -\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - 2\alpha_5$,

$$e_2(\nu, J) = -1\ \boxed{}\ -1 \quad 0\ \boxed{}\ 0 \qquad 0\ \boxed{}\ 1 \qquad 0\ \boxed{}\ -1 \quad -3\ \boxed{}\ -1$$

and

$$f_2(\nu, J) = -1\ \boxed{}\ -1 \quad -4\ \boxed{}\ -2 \qquad 0\ \boxed{}\ 1 \qquad 0\ \boxed{}\ -1 \quad -3\ \boxed{}\ -1$$

Lemma 3.3. *The set* $\mathrm{RC}(\infty)$ *is an abstract* $U_q(\mathfrak{g})$-*crystal with Kashiwara operators* e_a *and* f_a' *and remaining crystal structure given in Eq.* *(3.1).*

Proof. This proof here is similar to that given in Proposition 3.5 of [37]. We need to show the following, for (ν, J) in $\mathrm{RC}(\infty)$.

(1) If $e_a(\nu, J) \neq 0$ for $a \in I$, then $f_a' e_a(\nu, J) = (\nu, J)$.
(2) For any $a \in I$, we have $e_a f_a'(\nu, J) = (\nu, J)$.
(3) If $e_a(\nu, J) \neq 0$ for $a \in I$, then $\mathrm{wt}\big(e_a(\nu, J)\big) = \mathrm{wt}(\nu, J) + \alpha_a$.
(4) For $a \in I$, $\mathrm{wt}\big(f_a'(\nu, J)\big) = \mathrm{wt}(\nu, J) - \alpha_a$.
(5) For $a \in I$, $\varepsilon_a\big(f_a'(\nu, J)\big) = \varepsilon_a(\nu, J) + 1$ and $\varphi_a\big(f_a'(\nu, J)\big) = \varphi_a(\nu, J) - 1$.

Let (ν, J) be an arbitrary rigged configuration in $\mathrm{RC}(\infty)$. In what follows, we will suppose that $m_i^{(a)}$ is the number of parts of length i in the partition $\nu^{(a)}$ and that x is the smallest label of $(\nu, J)^{(a)}$. Set $(\nu', J') = f_a'(\nu, J)$ and $(\nu'', J'') = e_a(\nu, J)$.

To prove (1), suppose that (ν'', J'') is obtained from (ν, J) by changing the string (ℓ, x) to $(\ell - 1, x + 1)$, so that ℓ is the minimal length string among all strings with

label x. If $i < \ell$ and (i, y) is a string in (ν, J), then $p_i^{(a)}(\nu'') = p_i^{(a)}(\nu)$ by (2.4). Thus (i, y) is unaffected by the action of e_a, and $y \geq x + 1$. On the other hand, if $i \geq \ell$, then $p_i^{(a)}(\nu'') = p_i^{(a)}(\nu) + 2$ by (2.4). Thus (i, y) is replaced by the string $(i, y + 2)$ under the action of e_a and $y + 2 > x + 1$. In both cases, $(\ell - 1, x + 1)$ is the string with minimal label and longest length, so f_a' will change $(\ell - 1, x + 1)$ to (ℓ, x) and $f_a' e_a(\nu, J) = (\nu, J)$, as required.

Suppose that (ν', J') is obtained from (ν, J) by changing the string (ℓ, x) to $(\ell + 1, x - 1)$, so ℓ is the maximal length of all strings with label x. If $i \leq \ell$ and (i, y) is a string in $(\nu, J)^{(a)}$, then $p_i^{(a)}(\nu') = p_i^{(a)}(\nu)$ by (2.3). Thus (i, y) is left unaffected by the action of f_a', and $y > x - 1$ because x is the smallest label of (ν, J). On the other hand, if $i > \ell$, then $p_i^{(a)}(\nu') = p_i^{(a)}(\nu) - 2$ by (2.3). Thus (i, y) is replaced by $(i, y - 2)$ by the action of f_a' and $y - 2 \geq x - 1$. In both cases, $(\ell + 1, x - 1)$ is the string with minimal label and shortest length, so e_a will change $(\ell + 1, x - 1)$ to (ℓ, x) and $e_a f_a'(\nu, J) = (\nu, J)$ to prove (2).

For (3), if $(\nu'', J'') \neq 0$ for some $a \in I$, then (ν'', J'') is obtained from (ν, J) by replacing the string (ℓ, x) with $(\ell - 1, x + 1)$, where ℓ is the minimal length of all strings in $(\nu, J)^{(a)}$ having label x. Then $|\nu''^{(a)}| = |\nu^{(a)}| - 1$ and the result follows.

To see (4), if $x > 0$, then the string $(1, -1)$ is added to $(\nu, J)^{(a)}$. Then $|\nu'^{(a)}| = |\nu^{(a)}| + 1$. On the other hand, if $x \leq 0$ and ℓ is the maximal length of all strings in $(\nu, J)^{(a)}$ with label x, then the string (ℓ, x) is replaced by the string $(\ell + 1, x - 1)$, so $|\nu'^{(a)}| = |\nu^{(a)}| + 1$. In both cases, the equality $|\nu'^{(a)}| = |\nu^{(a)}| + 1$ yields the desired result.

The first part of (5) follows immediately from the definition. To see $\varphi_a\big(f_a'(\nu, J)\big) = \varphi_a\big((\nu, J)\big) - 1$, we have

$$\varphi_a\big(f_a'(\nu, J)\big) = \langle h_a, \mathrm{wt}\big(f_a'(\nu, J)\big)\rangle + \varepsilon_a\big(f_a'(\nu, J)\big)$$
$$= \langle h_a, \mathrm{wt}(\nu, J)\rangle - \langle h_a, \alpha_a\rangle + \varepsilon_a(\nu, J) + 1$$
$$= \langle h_a, \mathrm{wt}(\nu, J)\rangle - 2 + \varepsilon_a(\nu, J) + 1$$
$$= \varphi_a(\nu, J) - 1. \qquad \square$$

Definition 3.4. For a weight λ, let $T_\lambda = \{t_\lambda\}$ be the abstract $U_q(\mathfrak{g})$-crystal with operations defined by

$$e_a t_\lambda = f_a t_\lambda = 0, \quad \varepsilon_a(t_\lambda) = \varphi_a(t_\lambda) = -\infty, \quad \mathrm{wt}(t_\lambda) = \lambda.$$

For any abstract $U_q(\mathfrak{g})$-crystal B, the tensor product $T_\lambda \otimes B$ has the same crystal graph as B, but with each weight shifted by λ (and appropriate modifications to ε_a and φ_a). Following [17], there is an embedding of crystals

$$I_{\lambda+\mu, \lambda} \colon T_{-\lambda} \otimes B(\lambda) \longhookrightarrow T_{-\lambda-\mu} \otimes B(\lambda + \mu)$$

which sends $t_{-\lambda} \otimes u_\lambda \mapsto t_{-\lambda-\mu} \otimes u_{\lambda+\mu}$ and commutes with e_a for each $a \in I$. Moreover, for any three dominant weights λ, μ, and ξ, we get a commutative diagram

$$T_{-\lambda} \otimes B(\lambda) \xrightarrow{\quad I_{\lambda+\mu,\lambda} \quad} T_{-\lambda-\mu} \otimes B(\lambda+\mu)$$

$$I_{\lambda+\mu+\xi,\lambda} \searrow \qquad\qquad \downarrow I_{\lambda+\mu+\xi,\lambda+\mu} \qquad\qquad (3.3)$$

$$T_{-\lambda-\mu-\xi} \otimes B(\lambda+\mu+\xi).$$

Using the order on dominant integral weights given by $\mu \leq \lambda$ if and only if $\lambda - \mu \in P^+$, the set $\{T_{-\lambda} \otimes B(\lambda)\}_{\lambda \in P^+}$ is a directed system.

Theorem 3.5. *(See [17].) We have* $B(\infty) = \varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes B(\lambda)$.

By Theorem 2.7, each $B(\lambda)$ is $U_q(\mathfrak{g})$-crystal isomorphic to the graph $\mathrm{RC}(\lambda)$ generated by a highest weight rigged configuration (ν, J) of weight λ in $\mathrm{RC}(L)$ and the Kashiwara operators e_a and f_a defined in Definition 2.5. Thus we have

$$\varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes B(\lambda) \cong \varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes \mathrm{RC}(\lambda).$$

Our goal is to complete the diagram

$$\varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes B(\lambda) \xrightarrow{\quad \cong \quad} \varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes \mathrm{RC}(\lambda)$$

$$\Big\| \qquad\qquad\qquad\qquad\qquad \Big\| \qquad\qquad (3.4)$$

$$B(\infty) \dashrightarrow \mathrm{RC}(\infty)$$

by proving that the dashed equality on the right side of the square is actually an equality among $U_q(\mathfrak{g})$-crystals. Then we may define an isomorphism along the bottom of the square by taking the composite map along the top of the diagram.

Lemma 3.6. *Let λ and μ be dominant integral weights, and let*

$$\widetilde{I}_{\lambda+\mu,\lambda} \colon T_{-\lambda} \otimes \mathrm{RC}(\lambda) \longrightarrow T_{-\lambda-\mu} \otimes \mathrm{RC}(\lambda+\mu)$$

by $t_{-\lambda} \otimes (\nu, J) \mapsto t_{-\lambda-\mu} \otimes (\nu', J')$, *where* $(\nu', J') = (\nu, J)$ *as rigged configurations but has vacancy numbers considered as an element of* $\mathrm{RC}(\lambda+\mu)$. *For* $(\nu, J) \in \mathrm{RC}(\lambda)$, *the image* (ν', J') *is valid in* $\mathrm{RC}(\lambda+\mu)$. *Moreover,* $\widetilde{I}_{\lambda+\mu,\lambda}$ *is a crystal embedding.*

Proof. Write $\lambda = \sum_{(a,i) \in \mathcal{H}} i L_i^{(a)} \Lambda_a$ and $\mu = \sum_{(a,i) \in \mathcal{H}} i K_i^{(a)} \Lambda_a$. Then

$$\begin{aligned}
p_i^{(a)}(\nu) &= \sum_{j \geq 1} \min(i,j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}} (\alpha_a | \alpha_b) \min(i,j) m_j^{(b)} \\
&\leq \sum_{j \geq 1} \min(i,j) \big(L_j^{(a)} + K_j^{(a)} \big) - \sum_{(b,j) \in \mathcal{H}} (\alpha_a | \alpha_b) \min(i,j) m_j^{(b)} \\
&= p_i^{(a)}(\nu').
\end{aligned}$$

Thus

$$\max J_i^{(a)} = \max J_i'^{(a)} \le p_i^{(a)}(\nu) \le p_i^{(a)}(\nu'),$$

for all $(a, i) \in \mathcal{H}$ such that $J_i^{(a)} \ne \emptyset$ (and hence $J_i'^{(a)} \ne \emptyset$). This proves that (ν', J') is valid so that $\widetilde{I}_{\lambda+\mu,\lambda}$ is well-defined. Moreover, we have

$$
\begin{aligned}
\mathrm{wt}&\big(t_{-\lambda-\mu} \otimes (\nu', J')\big) \\
&= -(\lambda + \mu) + \mathrm{wt}(\nu', J') \\
&= -\sum_{(a,i)\in\mathcal{H}} i\big(L_i^{(a)} + K_i^{(a)}\big)\Lambda_a + \sum_{(a,i)\in\mathcal{H}} i\big((L_i^{(a)} + K_i^{(a)})\Lambda_a - m_i^{(a)}\alpha_a\big) \\
&= -\sum_{(a,i)\in\mathcal{H}} iL_i^{(a)}\Lambda_a + \sum_{(a,i)\in\mathcal{H}} i\big(L_i^{(a)}\Lambda_a - m_i^{(a)}\alpha_a\big) \\
&= -\lambda + \mathrm{wt}(\nu, J) \\
&= \mathrm{wt}\big(t_{-\lambda} \otimes \mathrm{wt}(\nu, J)\big),
\end{aligned}
$$

which shows that $\widetilde{I}_{\lambda+\mu,\lambda}$ preserves the weight map. Since $\widetilde{I}_{\lambda+\mu,\lambda}$ is the identity on rigged configurations, we obtain that e_a commutes with $\widetilde{I}_{\lambda+\mu,\lambda}$ and $\widetilde{I}_{\lambda+\mu,\lambda}$ preserves ε_a, for all $a \in I$. Also, f_a commutes with $\widetilde{I}_{\lambda+\mu,\lambda}$ if $f_a(\nu, J) \ne 0$ because the map is the identity map on rigged configurations. Then

$$
\begin{aligned}
\varphi_a(\nu', J') &= \varepsilon_a(\nu', J') + \langle h_a, \mu + \lambda\rangle \\
&= \varepsilon_a(\nu, J) + \langle h_a, \lambda\rangle + \langle h_a, \mu\rangle \\
&= \varphi_a(\nu, J) + \langle h_a, \mu\rangle,
\end{aligned}
$$

so we have

$$
\begin{aligned}
\varphi_a\big(t_{-\lambda} \otimes (\nu, J)\big) &= \max\{-\infty, \varphi_a(\nu, J) + \langle h_a, -\lambda\rangle\} \\
&= \varphi_a(\nu, J) + \langle h_a, -\lambda\rangle \\
&= \varphi_a(\nu', J') + \langle h_a, -\lambda - \mu\rangle \\
&= \max\{-\infty, \varphi_a(\nu', J') + \langle h_a, -\lambda - \mu\rangle\} \\
&= \varphi_a\big(t_{-\lambda-\mu} \otimes (\nu', J')\big).
\end{aligned}
$$

Hence $\widetilde{I}_{\lambda+\mu,\lambda}$ is a crystal embedding. $\quad\square$

To complete the construction of a directed system of crystals of rigged configurations, we have the following lemma, which follows from a modification of the proof of Lemma 3.6.

Lemma 3.7. *For dominant integral weights λ, μ, and ξ, the diagram*

$$
\begin{array}{ccc}
T_{-\lambda} \otimes \mathrm{RC}(\lambda) & \xrightarrow{\;\widetilde{I}_{\lambda+\mu,\lambda}\;} & T_{-\lambda-\mu} \otimes \mathrm{RC}(\lambda+\mu) \\[2mm]
& \searrow^{\widetilde{I}_{\lambda+\mu+\xi,\lambda}} & \Big\downarrow{\scriptstyle \widetilde{I}_{\lambda+\mu+\xi,\lambda+\mu}} \\[2mm]
& & T_{-\lambda-\mu-\xi} \otimes \mathrm{RC}(\lambda+\mu+\xi).
\end{array}
$$

commutes.

Proof. Follows by repeated use of Lemma 3.6 and the fact that $\widetilde{I}_{-,-}$ is the identity on rigged configurations. □

Lemma 3.8. *We have $\mathrm{RC}(\infty) = \varinjlim_{\lambda \in P^+} T_{-\lambda} \otimes \mathrm{RC}(\lambda)$ as abstract $U_q(\mathfrak{g})$-crystals.*

Proof. By Lemmas 3.6 and 3.7, $\{T_{-\lambda} \otimes \mathrm{RC}(\lambda)\}_{\lambda \in P^+}$ forms a directed system. Let X denote the direct limit $\varinjlim T_{-\lambda} \otimes \mathrm{RC}(\lambda)$. Let $\Theta \colon X \longrightarrow \mathrm{RC}(\infty)$ be the identity map on rigged configurations; that is, for $x \in X$ such that $x = \widetilde{I}_\lambda(t_{-\lambda} \otimes (\nu, J))$, we have $\Theta(x) = (\nu, J)$. To make the setting clear, we will denote the Kashiwara operators on X by $\vec{e}_a$, $\vec{f}_a$, the Kashiwara operators on $\mathrm{RC}(\lambda)$ and $T_{-\lambda} \otimes \mathrm{RC}(\lambda)$ by e_a^λ, f_a^λ, and the Kashiwara operators on $\mathrm{RC}(\infty)$ by e_a, f_a'.

To see that Θ commutes with Kashiwara lowering operators, for $x \in X$ and λ such that $x = \widetilde{I}_\lambda\big(t_{-\lambda} \otimes (\nu, J)\big)$, we have

$$
\vec{f}_a x = \widetilde{I}_\lambda\big(f_a^\lambda(t_{-\lambda} \otimes (\nu, J))\big) = \widetilde{I}_\lambda\big(t_{-\lambda} \otimes f_a^\lambda(\nu, J)\big),
$$

where $t_{-\lambda} \otimes (\nu, J)$ satisfies the condition $f_a^\lambda(\nu, J) \neq 0$. Note that any such λ will suffice by the definition of the direct limit. Thus

$$
\Theta(\vec{f}_a x) = f_a^\lambda(\nu, J) = f_a'(\nu, J) = f_a'\Theta(x).
$$

The calculation involving the Kashiwara raising operators is similar. By the definition of the weight function, it is clear that Θ preserves weights. Moreover, Θ sends the highest weight vector of X to the highest weight vector $(\nu_\emptyset, J_\emptyset)$ of $\mathrm{RC}(\infty)$, so Θ is a bijection. □

Theorem 3.9. *Let $\mathfrak{g}$ be a Lie algebra of simply-laced finite type. Then there exists a $U_q(\mathfrak{g})$-crystal isomorphism $B(\infty) \cong \mathrm{RC}(\infty)$ which sends $u_\infty \mapsto (\nu_\emptyset, J_\emptyset)$.*

Proof. By Lemma 3.8, the dashed arrow on the right-hand of the square in (3.4) becomes an isomorphism of $U_q(\mathfrak{g})$-crystals, so we may construct an isomorphism by composing the maps along the outside of the square. □

Remark 3.10. From this point forward, we denote f'_a simply by f_a. This should not cause any confusion.

4. Extending Theorem 3.9 to arbitrary simply-laced Kac–Moody algebras

We show the convexity condition holds for general symmetrizable types.

Lemma 4.1. *Consider a rigged configuration* (ν, J). *Fix* $(a, i) \in \mathcal{H}$ *and suppose that* $m_i^{(a)} = 0$. *Let* $C_{a,b}, C'_{a,b}, C^\vee_{a,b} \in \mathbf{Z}_{>0}$ *for all* $a, b \in I$, *and consider the generalization of the vacancy numbers for* (ν, J) *to*

$$p_i^{(a)} = \sum_{j \geq 1} \min(i, j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}} C^\vee_{a,b} A_{ab} \min(C_{a,b} i, C'_{a,b} j) m_j^{(b)}$$

We have

$$2p_i^{(a)} \geq p_{i-1}^{(a)} + p_{i+1}^{(a)}.$$

Proof. Consider any $(b, j) \in \mathcal{H}$ and define

$$Q_j^{(b)} = \sum_{k=1}^{\infty} \min(C_{a,b} j, C'_{a,b} k) m_k^{(b)}.$$

This is the number of boxes in the first $C_{a,b} j$ columns in the shape $C'_{a,b} \nu^{(b)}$. Set $\Theta_j^{(b)} = Q_j^{(b)} - Q_{j-1}^{(b)}$ and $\Xi_j^{(b)} = Q_{j+1}^{(b)} - Q_j^{(b)}$. We must have $\Theta_j^{(b)} \geq \Xi_j^{(b)} \geq 0$ since $C'_{a,b} \nu^{(b)}$ is a partition. Thus

$$2Q_j^{(b)} = 2Q_{j-1}^{(b)} + 2\Theta_j^{(b)}$$
$$\geq 2Q_{j-1}^{(b)} + \Theta_j^{(b)} + \Xi_j^{(b)}$$
$$= Q_{j-1}^{(b)} + Q_j^{(b)} + \Xi_j^{(b)}$$
$$= Q_{j-1}^{(b)} + Q_{j+1}^{(b)}.$$

Since $m_i^{(a)} = 0$, we have $\Xi_i^{(a)} = \Theta_i^{(a)}$, and so

$$2Q_i^{(a)} = Q_{i-1}^{(a)} + Q_{i+1}^{(a)}.$$

Recall $A_{ab} \leq 0$ for all $a \neq b$ and $C^\vee_{a,b} > 0$. Therefore we have $-C^\vee_{a,b} A_{ab} \geq 0$ for all $a \neq b$, and hence

$$-2 \sum_{(b,j) \in \mathcal{H}} C^\vee_{a,b} A_{ab} Q_j^{(b)} \geq - \sum_{(b,j) \in \mathcal{H}} C^\vee_{a,b} A_{ab} \big(Q_{j-1}^{(b)} + Q_{j+1}^{(b)} \big)$$

Similarly we can show that

$$\sum_{j>0} \min(i,j) L_j^{(a)} \geq \sum_{j>0} \min(i-1,j) L_j^{(a)} + \min(i+1,j) L_j^{(a)},$$

and hence

$$2p_i^{(a)} \geq p_{i-1}^{(a)} + p_{i+1}^{(a)}. \qquad \square$$

We also show the following proposition for generalized types.

Proposition 4.2. *Consider a rigged configuration* $(\nu, J) \in \mathrm{RC}(\infty)$. *Fix some* $a \in I$ *and consider the generalization of the vacancy numbers given in Lemma 4.1 such that* $p_\infty^{(a)} = \langle h_a, \mathrm{wt}(\nu, J) \rangle$. *Let* x *be the smallest label of* $(\nu, J)^{(a)}$ *and* $s = \min(0, x)$. *Then we have*

$$\varepsilon_a(\nu, J) = -s, \qquad \varphi_a(\nu, J) = p_\infty^{(a)} - s.$$

Proof. The proof that $\varphi_a(\nu, J) = p_\infty^{(a)} - s$ follows that given in [38, Lemma 3.6] and relies on the convexity statement of Lemma 4.1. The statement for $\varepsilon_a(\nu, J)$ follows from $p_\infty^{(a)} = \langle h_a, \mathrm{wt}(\nu, J) \rangle = \varphi_a(\nu, J) - \varepsilon_a(\nu, J)$ (or [37, Thm. 3.8]). $\quad \square$

Note that the proof of Theorem 2.7 given in [38] is based on the Stembridge axioms [43] and does not use the condition that the crystal of finite type. However it does rely upon Proposition 4.2 for simply-laced types (this is contained in [38]). Hence the proof holds for arbitrary simply-laced types, and it gives a rigged configuration model for highest weight modules in arbitrary simply-laced types. Similarly, the proof of Theorem 3.9 does not use any assumption that the Kac–Moody algebra be of finite type, so our result extends to arbitrary simply-laced types.

Theorem 4.3. *Let* $\mathfrak{g}$ *be of simply-laced type. Then there exists a* $U_q(\mathfrak{g})$-*crystal isomorphism* $B(\infty) \cong \mathrm{RC}(\infty)$ *which sends* $u_\infty \mapsto (\nu_\emptyset, J_\emptyset)$.

Example 4.4. Consider the hyperbolic Kac–Moody algebra $H_1^{(4)}$ (see [5] for the notation and list of Dynkin diagrams), whose Dynkin diagram is the complete graph on four vertices.
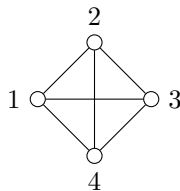
Then the partitions are enumerated as $(\nu^{(1)}, \nu^{(2)}, \nu^{(3)}, \nu^{(4)})$ and

Table 5.1
Well-known embeddings $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ of affine Kac–Moody algebras by type as given in [11] ($n \neq 1$).

type of $\mathfrak{g}$	$C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)}$	$B_n^{(1)}, A_{2n-1}^{(2)}$	$E_6^{(2)}, F_4^{(1)}$	$G_2^{(1)}, D_4^{(3)}$
type of $\widehat{\mathfrak{g}}$	$A_{2n-1}^{(1)}$	$D_{n+1}^{(1)}$	$E_6^{(1)}$	$D_4^{(1)}$

$$f_4 f_2^2 f_1 f_3 f_4^3 f_2 f_1 (\nu_\emptyset, J_\emptyset) = \begin{smallmatrix} 1 & \square & 2 \\ 1 & \square & 1 \end{smallmatrix} \qquad \begin{smallmatrix} 0 & \square\square & -1 \\ 1 & \square & 1 \end{smallmatrix} \qquad \begin{smallmatrix} 4 & \square & 2 \end{smallmatrix} \qquad \begin{smallmatrix} -2 & \square\square\square & -1 \\ 1 & & -1 \end{smallmatrix} .$$

5. Extending Theorem 3.9 to non-simply-laced Lie algebras

5.1. Virtual crystals

In this section, $\mathfrak{g}$ denotes an affine Kac–Moody algebra with classical subalgebra $\mathfrak{g}_0$. Fix one of the embeddings $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ from Table 5.1, so that $\widehat{\mathfrak{g}}$ is simply-laced with index set denoted by $\widehat{I}$. Let Γ be the Dynkin diagram of $\mathfrak{g}$ and $\widehat{\Gamma}$ be the Dynkin diagram of $\widehat{\mathfrak{g}}$.[2] These embeddings arise from the diagram foldings $\phi \colon \widehat{\Gamma} \searrow \Gamma$. We also have to define additional data $\gamma = (\gamma_a)_{a \in I}$ in the following way.

(1) Suppose Γ has a unique arrow. Removing the edge with this unique arrow leaves two connected components.
   (a) Suppose the arrow points towards the component of the special node 0. Then $\gamma_a = 1$ for all $a \in I$.
   (b) Suppose instead the arrow points away from the component of the special node 0. Then γ_a is the order of ϕ for all a in the component of 0 after removing the arrow. For a in the component not containing 0, set $\gamma_a = 1$.
(2) If Γ has two arrows, then Γ embeds into the Dynkin diagram of $A_{2n-1}^{(1)}$. Then $\gamma_a = 1$ for all $1 \leq a \leq n-1$, and for $a \in \{0, n\}$, we have $\gamma_a = 2$ if the arrow points away from a and $\gamma_a = 1$ otherwise.

We have two special cases of the above for types $A_1^{(1)}$ and $A_2^{(2)}$. For type $A_1^{(1)}$, we consider the diagram folding of $A_3^{(1)}$ given by $\phi^{-1}(0) = \{0, 2\}$ and $\phi^{-1}(1) = \{1, 3\}$ and $\gamma_0 = \gamma_1 = 1$. For type $A_2^{(2)}$, we consider the diagram folding of $D_4^{(1)}$ given by $\phi^{-1}(0) = \{0, 1, 3, 4\}$ and $\phi^{-1}(1) = \{2\}$ and $\gamma_0 = 1$ and $\gamma_1 = 4$.

The embeddings in Table 5.1 yield natural embeddings $\Psi \colon P \longrightarrow \widehat{P}$ of weight lattices as

$$\Lambda_a \mapsto \gamma_a \sum_{b \in \phi^{-1}(a)} \widehat{\Lambda}_b \qquad \text{and} \qquad \alpha_a \mapsto \gamma_a \sum_{b \in \phi^{-1}(a)} \widehat{\alpha}_b.$$

---

[2] From now on, if S is an object associated with $\mathfrak{g}$, then $\widehat{S}$ will denote the corresponding object associated with $\widehat{\mathfrak{g}}$ under the appropriate embedding listed above.

This implies that $\Psi(\delta) = c_0 \gamma_0 \widehat{\delta}$, where δ (resp. $\widehat{\delta}$) is the minimal positive imaginary root in P (resp. $\widehat{P}$).

Remark 5.1. There is another folding of $D_4^{(1)}$ to obtain $A_2^{(2)}$ by setting $\phi^{-1}(0) = \{2\}$ and $\phi^{-1}(1) = \{0, 1, 3, 4\}$, but with $\gamma_0 = \gamma_1 = 1$. Since $0 \notin \phi^{-1}(0)$, we have $\Psi(\delta) \neq c_0 \gamma_0 \widehat{\delta}$. This implies $\Psi(\delta) = c_{\phi(0)} \gamma_{\phi(0)} \widehat{\delta}$; i.e., we want the coefficients of $\widehat{\delta}$ to correspond to the image of 0 under the diagram folding. Alternatively we could consider this as a folding of $A_2^{(2)\dagger}$, which is the same as the Dynkin diagram of $A_2^{(2)}$ but with the labels of nodes interchanged (with 1 as the affine node).

Next we restrict our focus to untwisted types; that is, we only consider

$$C_n^{(1)} \longhookrightarrow A_{2n-1}^{(1)}, \qquad B_n^{(1)} \longhookrightarrow D_{n+1}^{(1)},$$
$$F_4^{(1)} \longhookrightarrow E_6^{(1)}, \qquad G_2^{(1)} \longhookrightarrow D_4^{(1)}. \tag{5.1}$$

When restricting to the classical subalgebras from (5.1), we get the embeddings

$$C_n \longhookrightarrow A_{2n-1}, \qquad B_n \longhookrightarrow D_{n+1},$$
$$F_4 \longhookrightarrow E_6, \qquad G_2 \longhookrightarrow D_4, \tag{5.2}$$

via diagram foldings.

If $\mathfrak{g}_0 \longhookrightarrow \widehat{\mathfrak{g}}_0$ is one of the embeddings from (5.2), then it induces an injection $v \colon B(\lambda) \longhookrightarrow B(\widehat{\lambda})$ as sets, where $\Psi(\lambda) = \widehat{\lambda}$. However, there is additional structure on the image under v as a *virtual crystal*, where e_a and f_a are defined on the image as

$$e_a^v = \prod_{b \in \phi^{-1}(a)} \widehat{e}_b^{\,\gamma_a} \qquad \text{and} \qquad f_a^v = \prod_{b \in \phi^{-1}(a)} \widehat{f}_b^{\,\gamma_a}, \tag{5.3}$$

respectively, and they commute with v [1,33,34]. These are known as the *virtual Kashiwara (crystal) operators*. It is shown in [16] that for any $a \in I$ and $b, b' \in \phi^{-1}(a)$ we have $e_b e_{b'} = e_{b'} e_b$ and $f_b f_{b'} = f_{b'} f_b$ as operators (recall that b and b' are not connected), so both e_a^v and f_a^v are well-defined. The inclusion map v also satisfies the following commutative diagram.

$$
\begin{array}{ccc}
B(\lambda) & \stackrel{v}{\longhookrightarrow} & B(\widehat{\lambda}) \\
{\scriptstyle \mathrm{wt}} \downarrow & & \downarrow {\scriptstyle \widehat{\mathrm{wt}}} \\
P & \stackrel{\hookleftarrow}{\underset{\Psi}{\longrightarrow}} & \widehat{P}
\end{array}
\tag{5.4}
$$

In [1], it was shown that this defines a $U_q(\mathfrak{g})$-crystal structure on the image of v. More generally, we define a virtual crystal as follows.

Definition 5.2. Consider any symmetrizable types $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ with index sets I and $\widehat{I}$, respectively. Let $\phi \colon \widehat{I} \longrightarrow I$ be a surjection such that b is not connected to b' for all $b, b' \in \phi^{-1}(a)$ and $a \in I$. Let $\widehat{B}$ be a $U_q(\widehat{\mathfrak{g}})$-crystal and $V \subseteq \widehat{B}$. Let $\gamma = (\gamma_a \in \mathbf{Z}_{>0} : a \in I)$. A *virtual crystal* is the quadruple $(V, \widehat{B}, \phi, \gamma)$ such that V has an abstract $U_q(\mathfrak{g})$-crystal structure defined using the Kashiwara operators e_a^v and f_a^v from (5.3) above,

$$\varepsilon_a := \gamma_a^{-1} \widehat{\varepsilon}_b, \qquad\qquad \varphi_a := \gamma_a^{-1} \widehat{\varphi}_b, \qquad\qquad \text{for all } b \in \phi^{-1}(a),$$

and $\mathrm{wt} := \Psi^{-1} \circ \widehat{\mathrm{wt}}$.

Remark 5.3. The definition of ε_a and φ_a forces all of our virtual crystals to be *aligned*, as defined in [33,34].

We say B *virtualizes* in $\widehat{B}$ if there exists a $U_q(\mathfrak{g})$-crystal isomorphism $v \colon B \longrightarrow V$. The resulting isomorphism is called the *virtualization map*. We denote the quadruple $(V, \widehat{B}, \phi, \gamma)$ simply by V when there's no risk of confusion.

The virtualization map v from rigged configurations of type $\mathfrak{g}_0$ to rigged configurations of type $\widehat{\mathfrak{g}}_0$ is defined by

$$\widehat{m}_{\gamma_a i}^{(b)} = m_i^{(a)}, \quad \widehat{J}_{\gamma_a i}^{(b)} = \gamma_a J_i^{(a)}, \tag{5.5}$$

for all $b \in \phi^{-1}(a)$. A $U_q(\mathfrak{g}_0)$-crystal structure on rigged configurations is defined by using virtual crystals [34]. Moreover, we use Eq. (5.5) to describe the virtual image of the type $\mathfrak{g}_0$ rigged configurations into type $\widehat{\mathfrak{g}}_0$ rigged configurations. Explicitly $(\widehat{\nu}, \widehat{J}) \in V$ if and only if

(1) $\widehat{m}_i^{(b)} = \widehat{m}_i^{(b')}$ and $\widehat{J}_i^{(b)} = \widehat{J}_i^{(b')}$ for all $b, b' \in \phi^{-1}(a)$,
(2) $\widehat{m}_i^{(b)} \in \gamma_a \mathbf{Z}$ and $\widehat{J}_i^{(b)} \in \gamma_a \mathbf{Z}$ for all $b \in \phi^{-1}(a)$, and
(3) $\widehat{m}_i^{(b)} = 0$ and $\widehat{J}_i^{(b)} = 0$ for all $j \notin \gamma_a \mathbf{Z}$ for all $b \in \phi^{-1}(a)$.

Example 5.4. Consider the rigged configuration in type C_2

$$1\,\square\,1 \qquad\quad \begin{array}{c} -1\,\square\,-1 \\ -1\,\square\,-1 \end{array}$$

with $L_1^{(1)} = L_1^{(2)} = 1$, all other $L_i^{(a)} = 0$, and weight $\Lambda_1 - \Lambda_2$. The corresponding virtual rigged configuration in type A_3 is

$$1\,\square\,1 \qquad \begin{array}{c} -2\,\square\square\,-2 \\ -2\,\square\square\,-2 \end{array} \qquad 1\,\square\,1$$

with $L_1^{(1)} = L_1^{(3)} = L_2^{(2)} = 1$, all other $L_i^{(a)} = 0$, and weight $\Lambda_1 + \Lambda_3 - 2\Lambda_2$.

Remark 5.5. There exist rigged configurations for $U_q(\mathfrak{g}_0)$-crystals when $\mathfrak{g}$ is of twisted affine type by considering $U_q'(\mathfrak{g})$ crystals; however, we omit those here in order to avoid confusion as we will be considering rigged configurations for $U_q(\mathfrak{g})$-crystals in the sequel. In particular, for type $A_{2n}^{(2)\dagger}$, the riggings of $\nu^{(n)}$ are in $\frac{1}{2}\mathbf{Z}$. See [34] for more information.

We note that it is sufficient to consider single tensor factors by the following proposition.

Proposition 5.6. *(See [33, Prop. 6.4].) Virtual crystals form a tensor category.*

Although [33] is concerned with $U_q'(\mathfrak{g})$-crystals, the proof of Proposition 5.6 does not use the $U_q'(\mathfrak{g})$-crystals condition, but instead is a statement about the tensor product rule. It has been cited as above in other papers; e.g., Proposition 3.3 of [34].

5.2. Extending Theorem 3.9 to all finite types

In this section we assume $\mathfrak{g}$ is of non-simply-laced finite type. For the vacancy numbers, we just consider this as the classical subcrystal in the corresponding untwisted affine type. Again, let $\mathrm{RC}(\infty)$ be the set generated by $(\nu_\emptyset, J_\emptyset)$ and e_a, f_a for $a \in I$, where e_a and f_a are defined as in Section 3.

Proposition 5.7 and Theorem 5.8 below are proven in [39] for all finite types. We will require these results in the sequel.

Proposition 5.7. *The crystal $\mathrm{RC}(\lambda)$ virtualizes in $\mathrm{RC}(\widehat{\lambda})$.*

Theorem 5.8. *Let $\mathfrak{g}$ be of finite type. We have $\mathrm{RC}(\lambda) \cong B(\lambda)$.*

Remark 5.9. Note the proof of Theorem 5.8 uses the fact that $B(\lambda)$ virtualizes in $B(\widehat{\lambda})$ in finite types [1,33,34].

By combining the virtualization results above with the method of proof given for Theorem 3.9, we may extend Theorem 3.9 to include non-simply-laced finite types.

Theorem 5.10. *Let $\mathfrak{g}$ be of any finite type. Then there exists a $U_q(\mathfrak{g})$-crystal isomorphism $\mathrm{RC}(\infty) \cong B(\infty)$ such that $(\nu_\emptyset, J_\emptyset) \mapsto u_\infty$.*

Proof. The proof of Theorem 3.9 holds here by following Section 3 and using Theorem 5.8 in place of Theorem 2.7. $\square$

We also have the virtualization of $B(\infty)$ crystals.

Proposition 5.11. *Let $\mathfrak{g}$ be of any finite type. The $U_q(\mathfrak{g})$-crystal $B(\infty)$ virtualizes in the $U_q(\widehat{\mathfrak{g}})$-crystal $\widehat{B}(\infty)$.*

Proof. This follows immediately from the fact that the diagram

$$
\begin{array}{ccc}
T_{-\lambda} \otimes B(\lambda) & \xrightarrow{\ I_{\lambda+\mu,\mu}\ } & T_{-\lambda-\mu} \otimes B(\lambda+\mu) \\
\downarrow & & \downarrow \\
T_{-\widehat{\lambda}} \otimes B(\widehat{\lambda}) & \xrightarrow[\ I_{\widehat{\lambda}+\widehat{\mu},\widehat{\mu}}\ ]{} & T_{-\widehat{\lambda}-\widehat{\mu}} \otimes B(\widehat{\lambda}+\widehat{\mu})
\end{array}
$$

commutes. □

5.3. Recognition theorem

From the above, we see that we only need to know the factors $(\gamma_a)_{a\in I}$ in order to show that we get a virtualization of the $U_q(\mathfrak{g})$-crystal of rigged configurations into a $U_q(\widehat{\mathfrak{g}})$-crystal by Eq. (5.5). Thus we make the following conjecture.

Conjecture 5.12. *Let $\mathfrak{g}$ be obtained via a diagram folding ϕ of a simply-laced type $\widehat{\mathfrak{g}}$. There exists $(\gamma_a)_{a\in I}$ such that $\mathrm{RC}(\lambda)$ virtualizes in $\mathrm{RC}(\widehat{\lambda})$ by Eq. (5.5).*

We have this for all finite and affine types using the foldings given in Table 5.1. We can also show this for all rank 2 with Cartan matrix

$$
\begin{pmatrix} 2 & x \\ y & 2 \end{pmatrix}
$$

by considering a diagram folding of $K_{x,y}$, the complete bipartite graph on x and y nodes, with $\gamma_1 = \gamma_2 = 1$. In such foldings, it is easy to see that Conjecture 5.12 holds from Eq. (2.2). In fact, we believe there exists a $\widehat{\mathfrak{g}}$ such that $\gamma_a = 1$ for all $a \in I$, and we call such a folding natural.

In their development of the geometric construction of the crystal basis, Kashiwara and Saito [19] established a recognition theorem for the crystal $B(\infty)$ valid for all symmetrizable Kac–Moody types. In this section, we will recall the recognition theorem with appropriate definitions and extend Theorem 3.9 to all Kac–Moody algebras satisfying Conjecture 5.12 using the recognition theorem.

Remark 5.13. A straightforward check shows that Proposition 5.7 holds in our affine setting, which requires Lemma 4.1, and so Conjecture 5.12 is true in affine types.

Remark 5.14. A priori, we do not have that $\mathrm{RC}(\lambda) \cong B(\lambda)$ for arbitrary symmetrizable types, as there is no equivalent version of Table 5.1 which would give the analogous statement to Theorem 5.8. Therefore we must change our techniques to show that the crystal $\mathrm{RC}(\infty) \cong B(\infty)$ by using the $B(\infty)$ recognition theorem given in [19]. Nevertheless, we will be able to show that those $\mathrm{RC}(\lambda)$ carved out of $\mathrm{RC}(\infty)$ are isomorphic to $B(\lambda)$ in Section 6.

From this viewpoint, it would be natural to restrict our attention for affine foldings from Table 5.1 given by

$$
\begin{aligned}
D_{n+1}^{(2)} &\longhookrightarrow A_{2n-1}^{(1)}, & A_{2n-1}^{(2)} &\longhookrightarrow D_{n+1}^{(1)}, \\
E_6^{(2)} &\longhookrightarrow E_6^{(1)}, & D_4^{(3)} &\longhookrightarrow D_4^{(1)},
\end{aligned}
\tag{5.6}
$$

as these foldings satisfy $\gamma_a = 1$ for all $a \in I$. The corresponding classical foldings from (5.2) are given by

$$
\begin{aligned}
B_n &\longhookrightarrow A_{2n-1}, & C_n &\longhookrightarrow D_{n+1}, \\
F_4 &\longhookrightarrow E_6, & G_2 &\longhookrightarrow D_4.
\end{aligned}
\tag{5.7}
$$

We should also note that we can get natural foldings of the other (non-degenerate) affine types by

$$
B_n^{(1)}, A_{2n}^{(2)} \longhookrightarrow D_{2n+1}^{(1)} \qquad C_n^{(1)} \longhookrightarrow D_{n+1}^{(1)}
$$

As in Remark 5.1, we have $\Psi(\delta) = c_{\phi(0)} \gamma_{\phi(0)} \widehat{\delta}$.

Definition 5.15. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and fix $a \in I$. Define $\mathbf{Z}_{(a)} = \{z_a(m) : m \in \mathbf{Z}\}$ with the abstract $U_q(\mathfrak{g})$-crystal structure given by

$$
\begin{aligned}
\mathrm{wt}\big(z_a(m)\big) = m\alpha_a, \quad \varphi_a\big(z_a(m)\big) &= m, \quad \varepsilon_a\big(z_a(m)\big) = -m, \\
\varphi_b\big(z_a(m)\big) = \varepsilon_b\big(z_a(m)\big) &= -\infty \text{ for } a \neq b, \\
e_a z_a(m) = z_a(m+1), \quad f_a z_a(m) &= z_a(m-1), \\
e_b z_a(m) = f_b z_a(m) &= 0 \text{ for } a \neq b.
\end{aligned}
$$

The crystal $\mathbf{Z}_{(a)}$ is called an *elementary crystal*.

Remark 5.16. The crystal $\mathbf{Z}_{(a)}$ was originally denoted by B_i in [15].

We must first prove a technical lemma about the virtual elementary crystals.

Lemma 5.17. *Let $\mathfrak{g}$ be a Kac–Moody algebra satisfying Conjecture 5.12. Let ϕ be the diagram folding with scaling factors $(\gamma_a)_{a \in I}$. Fix some $a \in I$. The elementary crystal $\mathbf{Z}_{(a)}$ virtualizes in $\widehat{\mathbf{Z}}_{(a)} = \bigotimes_{b \in \phi^{-1}(a)} \mathbf{Z}_{(b)}$ (for any order of the factors) with the virtualization map $v_{(a)}$ defined by*

$$
z_a(m) \mapsto \bigotimes_{b \in \phi^{-1}(a)} z_b(\gamma_a m).
$$

Proof. If $\widehat{\mathbf{Z}}_{(a)} = \mathbf{Z}_{(b)}$ where $\{b\} = \phi^{-1}(a)$, then it is easy to see the claim is true from Definition 5.15.

Now we assume $\widehat{\mathbf{Z}}_{(a)} = \mathbf{Z}_{(b_2)} \otimes \mathbf{Z}_{(b_1)}$ where $\{b_1, b_2\} = \phi^{-1}(a)$ and $b_1 \neq b_2$. If $b \notin \phi^{-1}(a)$, then

$$\varepsilon_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big) = \max(-\infty, -\infty - \langle h_b, \gamma_a m \alpha_{b_2}\rangle) = -\infty.$$

If $b = b_2$, then we have

$$\begin{aligned}
\varepsilon_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big) &= \max(-\gamma_a m, -\infty - \langle h_b, \gamma_a m \alpha_{b_2}\rangle) \\
&= -\gamma_a m \\
&= \gamma_a \varepsilon_a\big(z_a(m)\big)
\end{aligned}$$

since $-\infty + k = -\infty$ for any finite number k. If $b = b_1$, then we have

$$\begin{aligned}
\varepsilon_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big) &= \max(-\infty, -\gamma_a m - \langle h_b, \gamma_a m \alpha_{b_2}\rangle) \\
&= -\gamma_a m \\
&= \gamma_a \varepsilon_a\big(z_a(m)\big)
\end{aligned}$$

since $b_1 \neq b_2$. Similar statements hold for $\varphi_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big)$. From the tensor product rule,

$$\begin{aligned}
e_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big) &= \begin{cases} z_{b_2}(\gamma_a m) \otimes e_b\big(z_{b_1}(\gamma_a m)\big) & \text{if } b = b_1, \\ e_b\big(z_{b_2}(\gamma_a m)\big) \otimes z_{b_1}(\gamma_a m) & \text{if } b = b_2, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} z_{b_2}(\gamma_a m) \otimes z_{b_1}\big(\gamma_a(m+1)\big) & \text{if } b = b_1, \\ z_{b_2}\big(\gamma_a(m+1)\big) \otimes z_{b_1}(\gamma_a m) & \text{if } b = b_2, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
f_b\big(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\big) &= \begin{cases} z_{b_2}(\gamma_a m) \otimes f_b\big(z_{b_1}(\gamma_a m)\big) & \text{if } b = b_1, \\ f_b\big(z_{b_2}(\gamma_a m)\big) \otimes z_{b_1}(\gamma_a m) & \text{if } b = b_2, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} z_{b_2}(\gamma_a m) \otimes z_{b_1}\big(\gamma_a(m-1)\big) & \text{if } b = b_1, \\ z_{b_2}\big(\gamma_a(m-1)\big) \otimes z_{b_1}(\gamma_a m) & \text{if } b = b_2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus we have

$$
\begin{aligned}
\left(e_a^v \circ v\right)\left(z_a(m)\right) &= e_{b_1}^{\gamma_a} e_{b_2}^{\gamma_a}\left(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\right) \\
&= z_{b_2}\left(\gamma_a(m+1)\right) \otimes z_{b_1}\left(\gamma_a(m+1)\right) \\
&= v\left(z_a(m+1)\right) \\
&= v\left(e_a z_a(m)\right),
\end{aligned}
$$

and $\left(\widehat{e}_{a'} \circ v\right)\left(z_a(m)\right) = 0 = v\left(e_{a'} z_a(m)\right)$ for $a' \neq a$. Similar statements can be shown for f_a and $f_{a'}$ for $a' \neq a$. Lastly

$$
\mathrm{wt}\left(z_{b_2}(\gamma_a m) \otimes z_{b_1}(\gamma_a m)\right) = \gamma_a m(\alpha_{b_2} + \alpha_{b_1}) = \widehat{\mathrm{wt}}(z_a(m)).
$$

Therefore $\mathbf{Z}_{(a)}$ virtualizes in $\mathbf{Z}_{(b_2)} \otimes \mathbf{Z}_{(b_1)}$ with virtualization map v. It is clear that it is independent of the ordering. Moreover, we may generalize to the case of finitely many tensor factors using induction and associativity of the tensor product with a similar argument as above. □

Theorem 5.18 *(Recognition theorem). (See [19, Prop. 3.2.3].) Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, B be an abstract $U_q(\mathfrak{g})$-crystal, and x_0 be an element of B with weight zero. Assume the following conditions.*

(1) *$\mathrm{wt}(B) \subset Q_-$.*
(2) *x_0 is the unique element of B with weight zero.*
(3) *$\varepsilon_a(x_0) = 0$ for all $a \in I$.*
(4) *$\varepsilon_a(x) \in \mathbf{Z}$ for all $x \in B$ and $a \in I$.*
(5) *For every $a \in I$, there exists a strict crystal embedding $\Psi_a \colon B \longrightarrow \mathbf{Z}_{(a)} \otimes B$.*
(6) *$\Psi_a(B) \subset \{f_a^m z_a(0) : m \geq 0\} \times B$.*
(7) *For any $x \in B$ such that $x \neq x_0$, there exists $a \in I$ such that $\Psi_a(x) = f_a^m z_a(0) \otimes x'$ with $m > 0$ and $x' \in B$.*

Then B is isomorphic to $B(\infty)$.

Lemma 5.19. *Assume $\mathfrak{g}$ satisfies Conjecture 5.12. Then the crystal $\mathrm{RC}(\lambda)$ is generated by $(\nu_\emptyset, J_\emptyset)$ and f_a for all $a \in I$.*

Proof. By assumption, $\mathrm{RC}(\lambda)$ virtualizes in $\mathrm{RC}(\widehat{\lambda})$. Since $\mathrm{RC}(\widehat{\lambda}) \cong B(\widehat{\lambda})$ and $B(\widehat{\lambda})$ is generated by its highest weight vector and $\widehat{f}_a$ for all $a \in \widehat{I}$, the statement follows. □

Theorem 5.20. *Let $\mathfrak{g}$ be a Kac–Moody algebra satisfying Conjecture 5.12. Then $\mathrm{RC}(\infty) \cong B(\infty)$ as $U_q(\mathfrak{g})$-crystals.*

Proof. Let $\widehat{\mathrm{RC}}(\infty)$ denote the rigged configuration realization of the crystal $\widehat{B}(\infty)$ corresponding to the simply-laced Kac–Moody algebra $\widehat{\mathfrak{g}}$ coming from Theorem 4.3, so that

$$\widehat{\mathrm{RC}}(\infty) = \varinjlim_{\lambda \in P^+} \left(T_{-\lambda} \otimes \mathrm{RC}(\widehat{\lambda}) \right).$$

From Conjecture 5.12, we have

$$\mathrm{RC}(\infty) = \varinjlim_{\lambda \in P^+} \left(T_{-\lambda} \otimes \mathrm{RC}(\lambda) \right),$$

for reasons similar to the justification of Theorem 5.10. Hence $\mathrm{RC}(\infty)$ virtualizes in $\widehat{\mathrm{RC}}(\infty)$ as in Proposition 5.11. It remains to show that $\mathrm{RC}(\infty) \cong B(\infty)$ as $U_q(\mathfrak{g})$-crystals.

We note that (1) and (2) are satisfied from Eq. (3.1c) where $x_0 = (\nu_\emptyset, J_\emptyset)$. Condition (3) is satisfied directly by the definition of $(\nu_\emptyset, J_\emptyset)$, while (4) follows from the definition of ε_a on $\mathrm{RC}(\infty)$. The remaining properties require virtualization.

Let $v_{(a)}$ denote the virtualization map from Lemma 5.17. Now for each $a \in I$, define a crystal morphism $\Psi_a \colon \mathrm{RC}(\infty) \longrightarrow \mathbf{Z}_{(a)} \otimes \mathrm{RC}(\infty)$ in the following way. Consider the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{RC}(\infty) & \xrightarrow{\ \ v\ \ } & \widehat{\mathrm{RC}}(\infty) \\[2pt]
\Big\downarrow{\scriptstyle \Psi_a} & & \Big\downarrow{\scriptstyle \widehat{\Psi}_a} \\[6pt]
\mathbf{Z}_{(a)} \otimes \mathrm{RC}(\infty) & \xrightarrow{\ v_{(a)} \otimes v\ } & \widehat{\mathbf{Z}}_{(a)} \otimes \widehat{\mathrm{RC}}(\infty)
\end{array}
$$

Since both rows are virtualization maps by Proposition 5.6 and map on the right side is a strict embedding because $\widehat{\mathrm{RC}}(\infty) \cong \widehat{B}(\infty)$ by Theorem 4.3, we get a well-defined strict embedding $\Psi_a = (v_{(a)} \otimes v)^{-1} \circ \widehat{\Psi}_a \circ v$ for every $a \in I$.

For (6), notice the crystal $\mathrm{RC}(\infty)$ is generated from $(\nu_\emptyset, J_\emptyset)$ and f_a, for $a \in I$, from the direct limit characterization of $\mathrm{RC}(\infty)$ and Lemma 5.19. That is to say, we can write an arbitrary element (ν, J) of $\mathrm{RC}(\infty)$ as $(\nu, J) = f_{a_k} \cdots f_{a_1}(\nu_\emptyset, J_\emptyset)$ where $a_j \in I$. Since Ψ_a is strict and f_a^k is a nonzero operator on both $\mathbf{Z}_{(a)}$ and $\mathrm{RC}(\infty)$ for all $a \in I$ and $k \geq 0$, we have $\Psi_a\big(\mathrm{RC}(\infty)\big) \subset \{f_a^m z_a(0) : m \geq 0\} \times \mathrm{RC}(\infty)$.

Finally, set $(\nu, J) = f_{a_k} \cdots f_{a_1}(\nu_\emptyset, J_\emptyset)$ to be an arbitrary element of $\mathrm{RC}(\infty)$ and take $a = a_1$. Note that $\varphi_a(\nu_\emptyset, J_\emptyset) = 0$ by Eq. (3.1b). Then by the tensor product rule for crystals, we have $\Psi_a\big(f_a(\nu_\emptyset, J_\emptyset)\big) = f_a z_a(0) \otimes (\nu_\emptyset, J_\emptyset)$ because $\Psi_a(\nu_\emptyset, J_\emptyset) = z_a(0) \otimes (\nu_\emptyset, J_\emptyset)$. Therefore there exists some subsequence $(a_{j_1}, \ldots, a_{j_{k-m}})$ of $(a_1, \ldots, a_k)$ such that $a_1 = a_t$, for all $t \neq j_1, \ldots, j_{k-m}$, and $\Psi_a(\nu, J) = f_a^m z_a(0) \otimes f_{a_{j_t}} \cdots f_{a_{j_1}}(\nu_\emptyset, J_\emptyset)$ with $m > 0$. This shows condition (7), and we have $\mathrm{RC}(\infty) \cong B(\infty)$ by Theorem 5.18. $\square$

Open Problem 5.21. It would be interesting to find a proof which does not appeal to virtualization in order to prove (5), (6), and (7); in particular, to show that RC(∞) is generated only by $(\nu_\emptyset, J_\emptyset)$ and f_a, for all $a \in I$, without appealing to virtualization.

6. Projecting from RC(∞) to RC(λ)

The goal of this section is to show that taking valid rigged configurations is equivalent to projecting to highest weight $U_q(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is any symmetrizable Kac–Moody type satisfying Conjecture 5.12. Recall the one-element crystal $T_\lambda = \{t_\lambda\}$ given in Definition 3.4. Let $C = \{c\}$ be the one-element crystal with crystal operations defined by

$$\mathrm{wt}(c) = 0, \quad \varphi_a(c) = \varepsilon_a(c) = 0, \quad f_a(c) = e_a(c) = 0, \qquad a \in I.$$

It is known that the connected component in $C \otimes T_\lambda \otimes B(\infty)$ generated by $c \otimes t_\lambda \otimes u_\infty$ is isomorphic to $B(\lambda)$. In the setting of rigged configurations, recall that to pass from RC(∞) to RC(λ), we raise the weight by λ (equivalently we shift the vacancy numbers), which corresponds to tensoring with T_λ. Next we take only valid rigged configurations, and we will show that this restriction corresponds to tensoring with the crystal C.

Let $\mathrm{RC}_\lambda(\infty) = T_\lambda \otimes \mathrm{RC}(\infty)$ denote the crystal associated with the Verma module with highest weight λ. Strictly speaking,

$$\mathrm{RC}_\lambda(\infty) = \big\{ f_{a_k} \cdots f_{a_1} \big(t_\lambda \otimes (\nu_\emptyset, J_\emptyset) \big) : a_1, \ldots, a_k \in I, \ k \geq 0 \big\},$$

but by an abuse of notation, we will consider $\mathrm{RC}_\lambda(\infty)$ as the set of all rigged configurations generated by f_a ($a \in I$) from $(\nu_\emptyset, J_\emptyset)$ where the vacancy numbers and the weights are shifted by λ. That is, if $\lambda = \sum_{(a,i) \in \mathcal{H}} i L_i^{(a)} \Lambda_a$ is a dominant integral weight of type $\mathfrak{g}$, then for all $i \in \mathbf{Z}_{\geq 0}$ we have

$$p_i^{(a)}(\nu_\lambda) = \sum_{j \geq 0} \min(i, j) L_j^{(a)} + p_i^{(a)}(\nu), \qquad \mathrm{wt}(\nu_\lambda, J_\lambda) = \mathrm{wt}(\nu, J) + \lambda,$$

where $(\nu_\lambda, J_\lambda) \in \mathrm{RC}_\lambda(\infty)$ corresponds to $(\nu, J) \in \mathrm{RC}(\infty)$.

Theorem 6.1. *Let $\mathcal{C}_\emptyset$ denote the connected component of $C \otimes \mathrm{RC}_\lambda(\infty)$ generated by $c \otimes (\nu_\emptyset, J_\emptyset)$. The map $\Psi : \mathcal{C}_\emptyset \longrightarrow \mathrm{RC}(\lambda)$ sending $c \otimes (\nu_\lambda, J_\lambda) \mapsto (\nu_\lambda, J_\lambda)$ is a $U_q(\mathfrak{g})$-crystal isomorphism.*

Proof. Let $(\nu_\lambda, J_\lambda) \in \mathrm{RC}_\lambda(\infty)$ and $a \in I$. First,

$$\mathrm{wt}\big(c \otimes (\nu_\lambda, J_\lambda) \big) = \mathrm{wt}(c) + \mathrm{wt}(\nu_\lambda, J_\lambda) = \mathrm{wt}(\nu_\lambda, J_\lambda),$$

so Ψ preserves weights. Then,

$$\varepsilon_a\big(c \otimes (\nu_\lambda, J_\lambda)\big) = \max\big\{0, \varepsilon_a(\nu_\lambda, J_\lambda)\big\} = \varepsilon_a(\nu_\lambda, J_\lambda),$$

since $\varepsilon_a(\nu_\lambda, J_\lambda) \geq 0$, which implies that Ψ preserves ε_a. From the $\varepsilon_a\big(c \otimes (\nu_\lambda, J_\lambda)\big)$ computation above, we have

$$
\begin{aligned}
\varphi_a\big(c \otimes (\nu_\lambda, J_\lambda)\big) &= \max\big\{\varphi_a(\nu_\lambda, J_\lambda), \langle h_a, \mathrm{wt}(\nu_\lambda, J_\lambda)\rangle\big\} \\
&= \max\big\{\varepsilon_a(\nu_\lambda, J_\lambda) + \langle h_a, \mathrm{wt}(\nu_\lambda, J_\lambda)\rangle, \langle h_a, \mathrm{wt}(\nu_\lambda, J_\lambda)\rangle\big\} \\
&= \varepsilon_a(\nu_\lambda, J_\lambda) + \langle h_a, \mathrm{wt}(\nu_\lambda, J_\lambda)\rangle \\
&= \varphi_a(\nu_\lambda, J_\lambda).
\end{aligned}
$$

We have $\varphi_a(\nu_\lambda, J_\lambda) = 0$ if and only if $f_a(\nu_\lambda, J_\lambda) = 0$ in $\mathrm{RC}(\lambda)$ because $\mathrm{RC}(\lambda)$ is a (lower) regular crystal. Also if $\varphi_a(\nu_\lambda, J_\lambda) = 0$, we have

$$f_a\big(c \otimes (\nu_\lambda, J_\lambda)\big) = (f_a c) \otimes (\nu_\lambda, J_\lambda) = 0$$

by the tensor product rule. Similarly if $\varphi_a(\nu_\lambda, J_\lambda) > 0$, then

$$f_a\big(c \otimes (\nu_\lambda, J_\lambda)\big) = c \otimes f_a(\nu_\lambda, J_\lambda).$$

So $\Psi \circ f_a = f_a \circ \Psi$. Recall that $\varphi_a\big(c \otimes (\nu_\lambda, J_\lambda)\big) = \varphi_a(\nu_\lambda, J_\lambda) \geq 0$; so it follows, by the tensor product rule, that

$$\Psi\big(e_a\big(c \otimes (\nu_\lambda, J_\lambda)\big)\big) = \Psi\big(c \otimes e_a(\nu_\lambda, J_\lambda)\big) = e_a(\nu_\lambda, J_\lambda) = e_a \Psi\big(c \otimes (\nu_\lambda, J_\lambda)\big).$$

This completes the proof that Ψ is a crystal isomorphism. □

Thus, the projection map above corresponds to eliminating those rigged configurations which are not valid; that is, $\Psi(c \otimes (\nu, J)) = 0$ if (ν, J) is not valid. Therefore Theorem 5.20 implies the following.

Corollary 6.2. *Suppose Conjecture 5.12 holds, then we have* $\mathrm{RC}(\lambda) \cong B(\lambda)$.

Corollary 6.3. *Suppose Conjecture 5.12 holds, then the $U_q(\mathfrak{g})$-crystal $B(\lambda)$ virtualizes in the $U_q(\widehat{\mathfrak{g}})$-crystal $B(\widehat{\lambda})$.*

We also note that Proposition 4.2 extends to both $\mathrm{RC}(\infty)$ and $\mathrm{RC}(\lambda)$.

Example 6.4. Consider $\mathrm{RC}(\Lambda_0)$ with $\mathfrak{g} = A_2^{(1)}$. The top of the crystal graph is shown in Fig. 6.1.
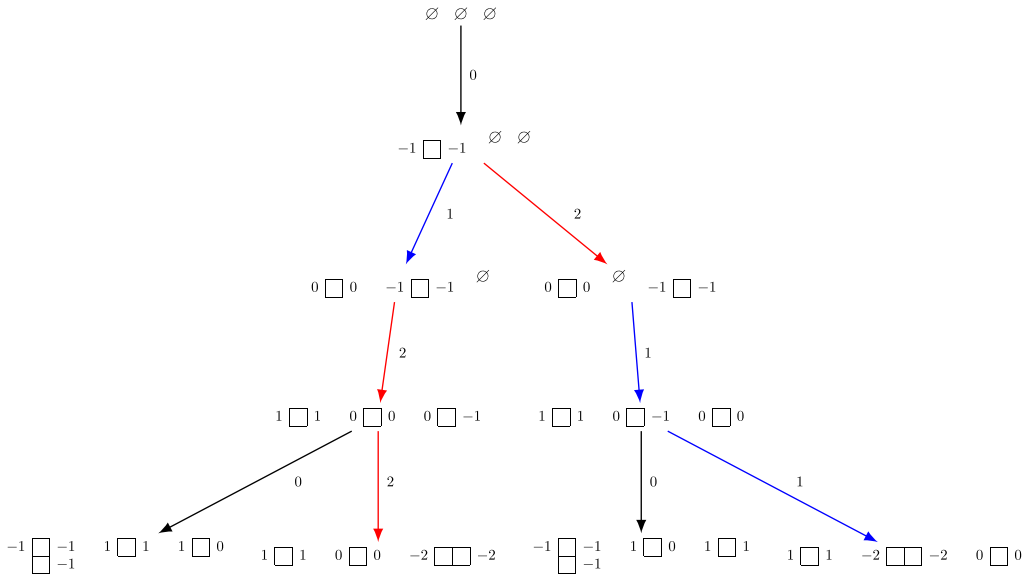
Fig. 6.1. The top of the crystal $RC(\Lambda_0)$ in type $A_2^{(1)}$, created using Sage.

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