Transformation Groups
Vol. 23, No. 2, 2018, pp. 501–525

COMBINATORIAL DESCRIPTIONS OF THE CRYSTAL STRUCTURE ON CERTAIN PBW BASES

BEN SALISBURY*

Department of
Mathematics
Central Michigan University
Mount Pleasant, MI, USA
ben.salisbury@cmich.edu

ADAM SCHULTZE**

Department of
Mathematics and Statistics
University at Albany
Albany, NY, USA
alschultze@albany.edu

PETER TINGLEY***

Department of
Mathematics and Statistics
Loyola University Chicago
Chicago, IL, USA
ptingley@luc.edu

Abstract. Using the theory of PBW bases, one can realize the crystal $B(\infty)$ for any semisimple Lie algebra over $\mathbf{C}$ using Kostant partitions as the underlying set. In fact there are many such realizations, one for each reduced expression for the longest element of the Weyl group. There is an algorithm to calculate the actions of the crystal operators, but it can be quite complicated. Here we show that, for certain reduced expressions, the crystal operators can also be described by a much simpler bracketing rule. We give conditions describing these reduced expressions, and show that there is at least one example in every type except possibly E_8, F_4 and G_2. We then discuss some examples.

1. Introduction

The crystal $B(\infty)$ of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbf{C}$ is a combinatorial object that contains a lot of information about $\mathfrak{g}$ and its finite-dimensional representations. Lusztig's early construction of canonical basis can be interpreted as giving a number of parameterizations of $B(\infty)$, one for each reduced expression for the longest element w_0 in the Weyl group (see [Lus10, Chaps. 41 and 42] or [Tin17]).

DOI: 10.1007/s00031-017-9434-9
*Partially supported by CMU Early Career grant #C62847 and by Simons Foundation grant #429950.
**Partially supported by NSF grant DMS-1265555.
***Partially supported by NSF grant DMS-1265555.
Received June 15, 2016. Accepted March 18, 2017.
Published online July 17, 2017.
Corresponding Author: Tingley, e-mail: ptingley@luc.edu.

In each of these realizations at least one of the crystal operators is very simple, but others may be complicated. However, Lusztig explicitly describes how the realizations are related for reduced expressions that differ by a braid move (see also Berenstein and Zelevinsky [BZ97] for the non-simply-laced cases). This gives a way to realize the whole crystal: an element is a PBW monomial with respect to some chosen reduced expression. To apply a crystal operator, modify the element via a sequence of braid moves until that operator is simple, then apply the operator, then modify it back.

This procedure is algorithmic, but can be complicated. In type A_n there is a simpler realization, using multisegments, where the crystal operators are given by a bracketing rule. As discussed in [CT15], this is naturally identified with Lusztig's crystal structure for the reduced expression

$$w_0 = (s_1 s_2 s_3 \cdots s_n)(s_1 \cdots s_{n-1}) \cdots (s_1 s_2) s_1.$$

Our main result is to generalize this by giving conditions on a reduced expression that ensure Lusztig's crystal structure is given by a similar rule, and to describe the resulting structure. For these words the crystal operators can be understood by combining many rank two calculations, and this combining is controlled by a bracketing procedure. There is at least one such reduced expression in every type except possibly E_8 and F_4, and we give detailed examples in the classical types. We do not consider type G_2 simply because the rank 2 calculation is more difficult. The method in [CT15] does not generalize easily outside of type A_n, and the proof here is quite different.

Reineke [Rei97] has also given explicit rules for calculating crystal operators on certain PBW monomials/Kostant partitions, which he does by identifying a Kostant partition with an isomorphism class of quiver representation. However, our method works in different generality: Reineke requires the reduced expression to be adapted to some orientation of the Dynkin diagram, while we require a "simply braided" reduced expression (see Definition 4.1). For example, the reduced expression we use for type D_n in §5.2 is not adapted to any orientation. Kwon [Kwo] recently showed that Reineke's structure does come from a bracketing rule in some cases (type A when the orientation has a single sink), and it would be interesting to compare that to the construction here.

The paper is organized as follows. §2 contains background material. §3 contains some examples of how crystal operators act on $B(\infty)$ as realized using PBW monomials. §4 contains our main results showing how, for certain reduced expressions, the crystal operators on PBW monomials can be calculated by bracketing rules. §5 contains some examples demonstrating these results.

Acknowledgements. We thank the developers of Sage [Dev16, SCc16], where many of the motivating calculations for this work were executed. We also thank Dinakar Muthiah for useful discussions. B. S. thanks Loyola University Chicago for its hospitality during a visit in which this manuscript began.

2. Background

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra over $\mathbf{C}$. Let I be the index set of $\mathfrak{g}$, $A = (a_{ij})$ the Cartan matrix, $\{\alpha_i\}_{i \in I}$ the positive simple roots, $\{\alpha_i^\vee\}_{i \in I}$ the simple coroots, Φ the set of roots, $\Phi^+ \subset \Phi$ the positive roots, P the weight lattice, $P^\vee$ the dual weight lattice, W the Weyl group with longest element w_0, and $\{s_i\}_{i \in I}$ the generating simple reflections. If $\mathfrak{g}$ is not simply-laced, let $\operatorname{diag}(d_i : i \in I)$ be the symmetrizing matrix of A. Let $(-|-)$ denote the symmetric, W-invariant bilinear form on P satisfying $(\alpha_i|\alpha_j) = d_i a_{ij}$ and let $\langle -, - \rangle \colon P^\vee \times P \to \mathbf{Z}$ be the canonical pairing. Denote the set of all reduced expressions for the longest element of the Weyl group by $R(w_0)$. Elements of $R(w_0)$ may also be referred to as reduced long words, or simply as reduced words. Enumerate the Dynkin diagram of $\mathfrak{g}$ following Bourbaki [Bou02] (see Figure 1).

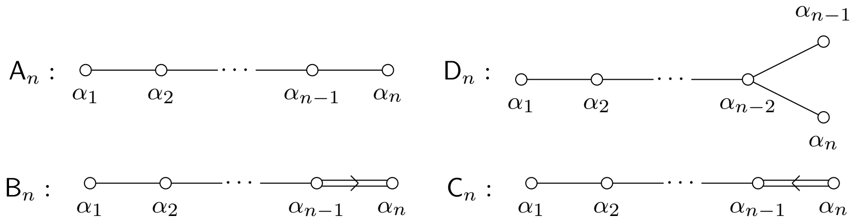


FIGURE 1. Dynkin diagrams of classical type following Bourbaki [Bou02].

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$, which is a $\mathbf{Q}(q)$-algebra generated by E_i, F_i, and q^h, for $i \in I$ and $h \in P^\vee$, subject to certain relations (see, for example, [HK02]). Let $U_q^-(\mathfrak{g})$ be the subalgebra generated by the F_i's. The star involution is the involutive $\mathbf{Q}(q)$-algebra antiautomorphism $* \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$E_i^* = E_i, \quad F_i^* = F_i, \quad (q^h)^* = q^{-h}.$$

2.1. Crystals

Let e_i, f_i be the Kashiwara operators on $U_q^-(\mathfrak{g})$ defined in [Kas91]. Let $\mathcal{A} \subset \mathbf{Q}(q)$ be the subring of functions regular at $q = 0$ and define $L(\infty)$ to be the $\mathcal{A}$-lattice spanned by

$$S = \{f_{i_1} f_{i_2} \cdots f_{i_t} \cdot 1 \in U_q^-(\mathfrak{g}) : t \geq 0,\ i_k \in I\}.$$

Let $e_i^* = * \circ e_i \circ *$ and $f_i^* = * \circ f_i \circ *$ be the operators twisted by the $*$-involution.

Theorem/Definition 2.1 ([Kas91]).

(i) *Let $\pi \colon L(\infty) \to L(\infty)/qL(\infty)$ be the natural projection and set $B(\infty) = \pi(S)$. Then $B(\infty)$ is a $\mathbf{Q}$-basis of $L(\infty)/qL(\infty)$.*
(ii) *For each $i \in I$ the operators e_i and f_i act on $L(\infty)/qL(\infty)$. Moreover, $e_i\big(B(\infty)\big) = B(\infty) \sqcup \{\mathbf{0}\}$ and $f_i\big(B(\infty)\big) \subset B(\infty)$.*
(iii) *The involution $*$ preserves $L(\infty)$ and $B(\infty)$. Hence e_i^* and f_i^* act on $B(\infty)$.*

For $i \in I$ and $b \in B(\infty)$, define

$$\varepsilon_i(b) = \max\{k \in \mathbf{Z}_{\geq 0} : e_i^k b \neq \mathbf{0}\}, \qquad \varepsilon_i^*(b) = \max\{k \in \mathbf{Z}_{\geq 0} : (e_i^*)^k b \neq \mathbf{0}\}.$$

Consider the weight map $\mathrm{wt}\colon B(\infty) \to P$ defined by

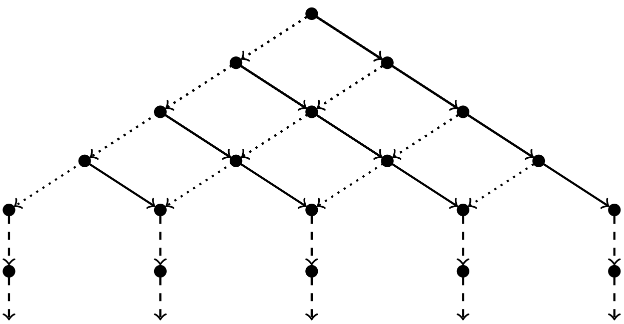
$$\mathrm{wt}(f_{i_1} f_{i_2} \cdots f_{i_t} \cdot 1) = -\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_t}.$$

The next proposition follows from [KS97, Prop. 3.2.3] (see also [TW16, Prop. 1.4]).

Proposition 2.2. *For all $b \in B(\infty)$ and all $i \neq j$ in I, we have*

(i) $f_i(b), f_i^*(b) \neq \mathbf{0}$,
(ii) $f_i^* f_j(b) = f_j f_i^*(b)$,
(iii) $\varepsilon_i(b) + \varepsilon_i^*(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \geq 0$,
(iv) $\varepsilon_i(b) + \varepsilon_i^*(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = 0$ *implies* $f_i(b) = f_i^*(b)$,
(v) $\varepsilon_i(b) + \varepsilon_i^*(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \geq 1$ *implies* $\varepsilon_i^*\big(f_i(b)\big) = \varepsilon_i^*(b)$ *and* $\varepsilon_i\big(f_i^*(b)\big) = \varepsilon_i(b)$,
(vi) $\varepsilon_i(b) + \varepsilon_i^*(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \geq 2$ *implies* $f_i f_i^*(b) = f_i^* f_i(b)$.

Corollary 2.3 ([TW16, Cor. 1.5])**.** *For any fixed elements $i \in I$ and $b \in B(\infty)$, the subset of $B(\infty)$ that can be reached from b by applying sequences of the operators e_i, f_i, e_i^*, f_i^* is of the following form, where the width of the diagram at the bottom is $\langle \alpha_i^\vee, \mathrm{wt}(b_{\mathrm{top}}) \rangle$ and b_{top} is the vertex at the top of the diagram. Here the width is 4.*



Here f_i acts on a vertex by following the solid or dashed arrow, and f_i^* acts by following the dotted or dashed arrow.

2.2. Reduced expressions and convex orders

Definition 2.4. A total order $\prec$ on Φ^+ is called *convex* if, for all triples of roots β, β', β'' with $\beta' = \beta + \beta''$, we have either $\beta \prec \beta' \prec \beta''$ or $\beta'' \prec \beta' \prec \beta$.

Theorem 2.5 ([Pap94])**.** *There is a bijection between $R(w_0)$ and convex orders on Φ^+: if $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, then the corresponding convex order $\prec$ is*

$$\beta_1 = \alpha_{i_1} \quad \prec \quad \beta_2 = s_{i_1}\alpha_{i_2} \quad \prec \quad \cdots \quad \prec \quad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}} \alpha_{i_N}.$$

By Theorem 2.5, we identity convex orderings with reduced expressions of w_0.

Lemma 2.6. *Fix two convex orders $\prec, \prec'$ on Φ^+ such that, for some root β,*

$$\{\alpha \in \Phi^+ : \alpha \prec \beta\} = \{\alpha \in \Phi^+ : \alpha \prec' \beta\}.$$

Call this set X. One can make a hybrid convex order by ordering X according to $\prec$ and $\Phi^+ \setminus X$ according to $\prec'$.

Proof. Let $|X| = k$, and consider the reduced words $\mathbf{i} = (i_1, \dots, i_k, i_{k+1}, \dots, i_N)$ and $\mathbf{i}' = (i'_1, \dots, i'_k, i'_{k+1}, \dots, i'_N)$ related to $\prec$ and $\prec'$ respectively as in Theorem 2.5. Then $\mathbf{i}'' = (i_1, \dots, i_k, i'_{k+1}, \dots, i'_N)$ is reduced and corresponds to the required convex order. $\square$

Definition 2.7. Fix an order $i_1, \dots, i_n$ on I. Define a corresponding order on Φ^+ as follows. For $\beta, \beta' \in \Phi^+$, let

$$\beta = \sum_{i \in I} p_i \alpha_i \quad \text{and} \quad \beta' = \sum_{i \in I} p'_i \alpha_i.$$

Then $\beta \prec \beta'$ if

(i) $\min\{k : p_{i_k} \neq 0\} < \min\{k : p'_{i_k} \neq 0\}$, or
(ii) $\min\{k : p_{i_k} \neq 0\} = \min\{k : p'_{i_k} \neq 0\}$, and, for that k,

$$\left(\frac{p_{i_{k+1}}}{p_{i_k}}, \frac{p_{i_{k+2}}}{p_{i_k}}, \dots, \frac{p_{i_n}}{p_{i_k}}\right) < \left(\frac{p'_{i_{k+1}}}{p'_{i_k}}, \frac{p'_{i_{k+2}}}{p'_{i_k}}, \dots, \frac{p'_{i_n}}{p'_{i_k}}\right)$$

in lexicographical order.

Example 2.8. Consider type D_4 and the enumeration $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, and $i_4 = 4$. The corresponding order on Φ^+ from Definition 2.7 is

$$1 \prec 12 \prec 124 \prec 123 \prec 1234 \prec 12234 \prec 2 \prec 24 \prec 23 \prec 234 \prec 3 \prec 4,$$

where, for example, 124 means $\alpha_1 + \alpha_2 + \alpha_4$. This is different from the convex order corresponding to this enumeration in, for example, [Lec04] (which orders roots $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ by lexicographically ordering the set $(c_1/c, c_2/c, c_3/c, c_4/c)$ where $c = c_1 + c_2 + c_3 + c_4$) since, in particular, the roots 124 and 123 are reversed.

Lemma 2.9. *For any enumeration of I, the order on Φ^+ from Definition 2.7 is convex.*

Proof. It is immediate that $\prec$ defines a total order on Φ^+. Fix $\beta = \sum_{i \in I} p_i \alpha_i$, $\beta' = \sum_{i \in I} p'_i \alpha_i$, and $\beta'' = \sum_{i \in I} p''_i \alpha_i$ such that $\beta' = \beta + \beta''$. Without loss of generality, assume $\beta \prec \beta''$. Let $m = \min\{k : p_{i_k} \neq 0\}$, and similarly for m' and m''.

If $m < m''$, then $m' = m < m''$, so $\beta' \prec \beta''$. Also, $p_{i_m} = p'_{i_m}$ and, for each $j > m$, $p'_{i_j} \geq p_{i_j}$, with at least one of these inequalities being strict, so $\beta \prec \beta'$.

If $m = m''$, then this is also equal to m'. Let $s > m$ be minimal such that $p_{i_s}/p_{i_m} < p''_{i_s}/p''_{i_m}$. Clearly

$$\frac{p_{i_s}}{p_{i_m}} < \frac{p'_{i_s}}{p'_{i_m}} < \frac{p''_{i_s}}{p''_{i_m}} \quad \text{and} \quad \frac{p_{i_j}}{p_{i_m}} = \frac{p'_{i_j}}{p'_{i_m}} = \frac{p''_{i_j}}{p''_{i_m}}$$

for all $m < j < s$, so $\beta \prec \beta' \prec \beta''$, as required. $\square$

2.3. PBW bases and enumerations of $B(\infty)$ by Kostant partitions

For $c \in \mathbf{Z}_{>0}$, define

$$F_i^{(c)} := \frac{F_i^c}{[c]!} \quad \text{where} \quad [c]! := \prod_{j=1}^{c} \frac{q^j - q^{-j}}{q - q^{-1}}.$$

Given $\mathbf{i} = (i_1, \dots, i_N) \in R(w_0)$ and $\boldsymbol{c} = (c^{\mathbf{i}}_\beta \in \mathbf{Z}^N_{\geq 0} : \beta \in \Phi^+)$, define

$$F_{\mathbf{i}}^{\boldsymbol{c}} = F_{\mathbf{i}:\beta_1}^{(c^{\mathbf{i}}_{\beta_1})} F_{\mathbf{i}:\beta_2}^{(c^{\mathbf{i}}_{\beta_2})} \cdots F_{\mathbf{i}:\beta_N}^{(c^{\mathbf{i}}_{\beta_N})} \quad \text{where} \quad F_{\mathbf{i}:\beta_k}^{(c^{\mathbf{i}}_{\beta_k})} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}} (F_{i_k}^{(c^{\mathbf{i}}_{\beta_k})}), \tag{1}$$

and T_i is the Lusztig automorphism of $U_q(\mathfrak{g})$ defined in [Lus10, Section 37.1.3] (there, it is denoted $T''_{i,-1}$). Then the set $\mathscr{B}_{\mathbf{i}} = \{F_{\mathbf{i}}^{\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{Z}^N_{\geq 0}\}$ forms a $\mathbf{Q}(q)$-basis of $U_q^-(\mathfrak{g})$, called the *PBW basis.* The notation β_k used in the subscript of $F_{\mathbf{i}:\beta_k}$ is because, for all k,

$$\mathrm{wt}(F_{\mathbf{i},\beta_k}) = -s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} = -\beta_k. \tag{2}$$

These are exactly the negative roots. We index the root vectors by the corresponding positive roots β_k. When the context is clear we omit the subscript $\mathbf{i}$ in $F_{\mathbf{i}:\beta_k}$.

Theorem 2.10 ([Sai94])**.** *For* $\mathbf{i} \in R(w_0)$*,* $\mathrm{Span}_{\mathcal{A}}(\mathscr{B}_{\mathbf{i}}) = L(\infty)$ *and* $\mathscr{B}_{\mathbf{i}} + qL(\infty) = B(\infty)$*.*

Definition 2.11. For $b \in B(\infty)$, the *Lusztig data* associated to b is the tuple $\boldsymbol{c}^{\mathbf{i}}(b) \in \mathbf{Z}^N_{\geq 0}$ such that $F_{\mathbf{i}}^{\mathbf{c}^{\mathbf{i}}(b)} + qL(\infty) = b$.

This gives a parametrization of $B(\infty)$ by $\mathbf{i}$-Lusztig data for any $\mathbf{i}$.

Proposition 2.12 ([BZ01, Lus10])**.** *Fix* $b \in B(\infty)$ *and* $i \in I$*.*

(i) *Let* $\mathbf{i} \in R(w_0)$ *be such that, in the corresponding convex order,* $\beta_1 = \alpha_i$*. If* $\mathbf{c}^{\mathbf{i}}(b) = (c_{\beta_1}, c_{\beta_2}, \dots, c_{\beta_N})$*, then* $\mathbf{c}^{\mathbf{i}}(f_i b) = (c_{\beta_1} + 1, c_{\beta_2}, \dots, c_{\beta_N})$*. If* $c_{\beta_1} = 0$ *then* $e_i b = \mathbf{0}$ *and otherwise* $\mathbf{c}^{\mathbf{i}}(e_i b) = (c_{\beta_1} - 1, c_{\beta_2}, \dots, c_{\beta_N})$*.*

(ii) *Let* $\mathbf{i} \in R(w_0)$ *be such that, in the corresponding convex order,* $\beta_N = \alpha_i$*. If* $\boldsymbol{c}^{\mathbf{i}}(b) = (c_1, c_2, \dots, c_N)$ *then* $\mathbf{c}^{\mathbf{i}}(f_i^* b) = (c_{\beta_1}, c_{\beta_2}, \dots, c_{\beta_N} + 1)$*. If* $c_{\beta_N} = 0$ *then* $e_i^* b = \mathbf{0}$ *and otherwise* $\mathbf{c}^{\mathbf{i}}(e_i^* b) = (c_{\beta_1}, c_{\beta_2}, \dots, c_{\beta_N} - 1)$*.*

In order to use Proposition 2.12 to understand the whole crystal structure on $B(\infty)$, we need to understand how the Lusztig data $\boldsymbol{c}^{\mathbf{i}}(b)$ are related for different $\mathbf{i}$. Since all reduced expressions are related by a sequence of braid moves, it is enough to understand what happens to $\boldsymbol{c}^{\mathbf{i}}(b)$ when $\mathbf{i}$ changes by a single braid move.

Lemma 2.13. *Fix* $\mathbf{i} \in R(w_0)$ *and let* $\{\beta_1 \prec \cdots \prec \beta_N\}$ *be the corresponding convex ordering of* Φ^+*.*

(i) *There is a reduced expression* $\mathbf{i}'$ *related to* $\mathbf{i}$ *by a braid move*

$$(i_k, i_{k+1}) \to (i_{k+1}, i_k)$$

if and only if $(\beta_k | \beta_{k+1}) = 0$*. In this case,* $\beta_k \prec \beta_{k+1}$ *is replaced by* $\beta_{k+1} \prec' \beta_k$ *after the braid move.*

(ii) *There is a reduced expression* $\mathbf{i}'$ *related to* $\mathbf{i}$ *by a braid move*

$$(i_k, i_{k+1}, i_{k+2}) \to (i_{k+1}, i_k, i_{k+1}),$$

with $i_k = i_{k+2}$*, if and only if* $\{\beta_k, \beta_{k+1}, \beta_{k+2}\}$ *form a root system of type* $\mathfrak{sl}_3$*. In this case,* $\beta_k \prec \beta_{k+1} \prec \beta_{k+2}$ *is replaced by* $\beta_{k+2} \prec' \beta_{k+1} \prec' \beta_k$ *after the braid move.*

(iii) *There is a reduced expression* $\mathbf{i}'$ *related to* $\mathbf{i}$ *by a braid move*

$$(i_k, i_{k+1}, i_{k+2}, i_{k+3}) \to (i_{k+1}, i_k, i_{k+1}, i_k),$$

with $i_k = i_{k+2}$ *and* $i_{k+1} = i_{k+3}$*, if and only if* $\{\beta_k, \beta_{k+1}, \beta_{k+2}, \beta_{k+3}\}$ *form a root system of type* B_2*. In this case,* $\beta_k \prec \beta_{k+1} \prec \beta_{k+2} \prec \beta_{k+3}$ *is replaced by* $\beta_{k+3} \prec' \beta_{k+2} \prec' \beta_{k+1} \prec' \beta_k$ *after the braid move.*

Proof. Let $w = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}$ and suppose $(\beta_k|\beta_{k+1}) = 0$. Then

$$\begin{aligned} 0 = (\beta_k|\beta_{k+1}) &= (w\alpha_{i_k}|ws_{i_k}\alpha_{i_{k+1}}) \\ &= (\alpha_{i_k}|s_{i_k}\alpha_{i_{k+1}}) \\ &= (s_{i_k}\alpha_{i_k}|\alpha_{i_{k+1}}) \\ &= -(\alpha_{i_k}|\alpha_{i_{k+1}}). \end{aligned}$$

Hence $a_{i_k,i_{k+1}} = 0$, and we can perform a 2-term braid move exactly when $a_{i_k,i_{k+1}} = 0$. Now $\beta_{k+1} = w\alpha_{i_{k+1}}$ and $\beta_k = ws_{i_{k+1}}\alpha_{i_k}$, so reversing i_k and i_{k+1} clearly reverses these.

The proof of (ii) is similar: both the condition that $\{\beta_k, \beta_{k+1}, \beta_{k+2}\}$ forms a root system of type $\mathfrak{sl}_3$ and the ability to perform a 3-term braid move of the desired form are equivalent to $a_{i_k,i_{k+1}} = -1$. In this situation, with w as above,

$$\begin{aligned} \beta_{k+2} &= ws_{i_k}s_{i_{k+1}}\alpha_{i_k} = ws_{i_k}(\alpha_{i_k} + \alpha_{i_{k+1}}) = w\alpha_{i_{k+1}}, \\ \beta_{k+1} &= ws_{i_k}\alpha_{i_{k+1}} = w(\alpha_{i_k} + \alpha_{i_{k+1}}) = ws_{i_{k+1}}\alpha_{i_k}, \\ \beta_k &= w\alpha_{i_k} = ws_{i_{k+1}}(\alpha_{i_k} + \alpha_{i_{k+1}}) = ws_{i_{k+1}}s_{i_k}\alpha_{i_{k+1}}. \end{aligned}$$

By Theorem 2.5, we have $\beta_{k+2} \prec' \beta_{k+1} \prec' \beta_k$.

The proof of (iii) is also similar. □

Lemma 2.14 ([Lus90, §2.1])**.** *Let* $b \in B(\infty)$*.*

(i) *If* $\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$*, then the Lusztig data* $\boldsymbol{c}(b)$ *and* $\boldsymbol{c}'(b)$ *with respect to the two possible reduced expressions are identical (meaning* $c_{\alpha_1} = c'_{\alpha_1}$ *and* $c_{\alpha_2} = c'_{\alpha_2}$*, but the order is reversed).*

(ii) *If* $\mathfrak{g} = \mathfrak{sl}_3$*, then the Lusztig data* $\boldsymbol{c}(b)$ *and* $\boldsymbol{c}'(b)$ *with respect to the two possible reduced expressions are related by*

$$\begin{aligned} c'_{\alpha_1} &= \max\{c_{\alpha_1+\alpha_2}, c_{\alpha_1} + c_{\alpha_1+\alpha_2} - c_{\alpha_2}\}, \\ c'_{\alpha_1+\alpha_2} &= \min\{c_{\alpha_1}, c_{\alpha_2}\}, \\ c'_{\alpha_2} &= \max\{c_{\alpha_1+\alpha_2}, c_{\alpha_2} + c_{\alpha_1+\alpha_2} - c_{\alpha_1}\}. \end{aligned}$$

More generally, if a reduced expression is changed by a 2- *or* 3-*term braid move, the Lusztig data changes according to these rules for the affected roots.*

Lemma 2.15 ([BZ97, Thm. 3.1])**.** *If $\mathfrak{g} = \mathsf{B}_2$, with α_1 being the long simple root, and $\boldsymbol{c}(b)$ is the Lusztig data of $b \in B(\infty)$ with respect to one reduced expression, then the Lusztig data $\mathbf{c}'(b)$ with respect to the other reduced expression is*

$$\begin{aligned} c'_{\alpha_1} &= c_{\alpha_1} + c_{\alpha_1+\alpha_2} + c_{\alpha_1+2\alpha_2} - \pi_1, \\ c'_{\alpha_1+\alpha_2} &= 2\pi_1 - \pi_2, \\ c'_{\alpha_1+2\alpha_2} &= \pi_2 - \pi_1, \\ c'_{\alpha_2} &= c_{\alpha_1+\alpha_2} + 2c_{\alpha_1+2\alpha_2} + c_{\alpha_2} - \pi_2, \end{aligned}$$

where

$$\begin{aligned} \pi_1 &= \min\{c_{\alpha_1} + c_{\alpha_1+\alpha_2}, c_{\alpha_1} + c_{\alpha_2}, c_{\alpha_1+2\alpha_2} + c_{\alpha_2}\}, \\ \pi_2 &= \min\{2c_{\alpha_1} + c_{\alpha_1+\alpha_2}, 2c_{\alpha_1} + c_{\alpha_2}, 2c_{\alpha_1+2\alpha_2} + c_{\alpha_2}\}. \end{aligned}$$

More generally, if a reduced expression is changed by a 2- or 3-term braid move, the Lusztig data changes according to these rules for the affected roots.

The following is equivalent to Lemma 2.15, but is more suited to our purposes.

Lemma 2.16. *Let $\mathfrak{g} = \mathsf{B}_2$, with α_1 the long root and α_2 the short root. If $\boldsymbol{c}(b) = (c_{\alpha_1}, c_{\alpha_1+\alpha_2}, c_{\alpha_1+2\alpha_2}, c_{\alpha_2})$ is the Lusztig data for $b \in B(\infty)$ with respect to one reduced expression, then the Lusztig data $\mathbf{c}'(b)$ with respect to the other reduced expression can be found as follows: Consider the string of large and small brackets*

$$T(\boldsymbol{c}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}} \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1}} \underbrace{\Big)\cdots\Big)}_{c_{\alpha_1+2\alpha_2}} \underbrace{(\cdots(}_{c_{\alpha_1+\alpha_2}} \underbrace{)\cdots)}_{c_{\alpha_2}} \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1+2\alpha_2}}.$$

Cancel brackets using the rules:

- *cancel as many pairs* $()$ *and* $\Big(\Big)$ *as possible, and*
- *cancel remaining "$\Big($" and "$)$" using the rule that one "$\Big($" can cancel either one or two "$)$", and as many total brackets as possible are canceled.*

Then

$$\begin{aligned} c'_{\alpha_1} &= \#\, \textit{uncanceled}\ (\ + \ \#\, \textit{uncanceled}\ \Big(\\ c'_{\alpha_1+\alpha_2} &= \#\, \textit{canceled}\ () \ + \ \#\, \textit{canceled}\ \Big() \\ c'_{\alpha_1+2\alpha_2} &= \#\, \textit{canceled}\ \Big(\Big) \ + \ \#\, \textit{canceled}\ \Big()) \\ c'_{\alpha_2} &= \#\, \textit{uncanceled}\ ) \ + \ 2\#\, \textit{uncanceled}\ \Big). \end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned} c'_{\alpha_1} &= \max\{c_{\alpha_1+2\alpha_2}, c_{\alpha_1+\alpha_2} + c_{\alpha_1+2\alpha_2} - c_{\alpha_2}, c_{\alpha_1} + c_{\alpha_1+\alpha_2} - c_{\alpha_2}\} \\ c'_{\alpha_1+\alpha_2} &= \min\{\max\{c_{\alpha_1+\alpha_2}, c_{\alpha_1} + c_{\alpha_1+\alpha_2} - c_{\alpha_1+2\alpha_2}\}, c_{\alpha_2}\} \\ c'_{\alpha_1+2\alpha_2} &= \min\{\max\left\{c_{\alpha_1+2\alpha_2}, c_{\alpha_2} + 2c_{\alpha_1+2\alpha_2} - c_{\alpha_1+\alpha_2} - c_{\alpha_1}\right\}, c_{\alpha_1}\} \\ c'_{\alpha_2} &= \max\{c_{\alpha_1+\alpha_2}, 2c_{\alpha_1+2\alpha_2} + c_{\alpha_1+\alpha_2} - 2c_{\alpha_1}, c_{\alpha_2} + 2c_{\alpha_1+2\alpha_2} - 2c_{\alpha_1}\}. \end{aligned} \tag{4}$$

More generally, if a reduced expression is changed by a 4-term braid move, then the Lusztig data changes according to these rules for the affected roots.

Proof. That (3) and (4) agree is straightforward. We must show that they agree with Lemma 2.15. The first and fourth formulas in (4) agree with the corresponding formulas in Lemma 2.15 by a straightforward calculation. Since $\alpha_1 + \alpha_2$ and $\alpha_1 + 2\alpha_2$ are linearly independent, it suffices to show that (3) gives data of the correct weight. So, let $\mathrm{wt}(b) = x\alpha_1 + y\alpha_2$. Clearly

$$x = \#(+ \#\Big(\quad \text{and} \quad y = \#) \; + \; 2\#\Big). \tag{5}$$

But then

$$\begin{aligned} c'_{\alpha_2} + 2c'_{\alpha_1+2\alpha_2} + c'_{\alpha_1+\alpha_2} &= \#) \; + \; 2\#\Big) = y, \text{ and} \\ c'_{\alpha_1} + c'_{\alpha_1+\alpha_2} + c'_{\alpha_1+2\alpha_2} &= \#(+ \#\Big(= x. \end{aligned} \tag{6}$$

So, $\mathrm{wt}(\boldsymbol{c}')$ is in fact correct. $\square$

Remark 2.17. The analogue of Lemma 2.16 in the case where $\mathfrak{g} = \mathsf{C}_2$, α_1 is the short root, and α_2 is the long root can be obtained by interchanging the roles of α_1 and α_2 and reversing the equalities in Equation (3); that is, if $\boldsymbol{c}(b) = (c_{\alpha_1}, c_{2\alpha_1+\alpha_2}, c_{\alpha_1+\alpha_2}, c_{\alpha_2})$ is the Lusztig data for $b \in B(\infty)$ with respect to one reduced expression, then the Lusztig data $\boldsymbol{c}'(b)$ with respect to the other reduced expression can be found using the bracketing sequence

$$T(\boldsymbol{c}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}} \;\; \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2}} \;\; \underbrace{\Big)\cdots\Big)}_{c_{2\alpha_1+\alpha_2}} \;\; \underbrace{(\cdots(}_{c_{\alpha_1+\alpha_2}} \;\; \underbrace{)\cdots)}_{c_{\alpha_1}} \;\; \underbrace{\Big(\cdots\Big(}_{c_{2\alpha_1+\alpha_2}}$$

and setting

$$\begin{aligned} c'_{\alpha_1} &= \#\,\text{uncanceled }) \; + \; 2\#\,\text{uncanceled } \Big) \\ c'_{2\alpha_1+\alpha_2} &= \#\,\text{canceled } \Big(\Big) \; + \; \#\,\text{canceled } \Big()) \\ c'_{\alpha_1+\alpha_2} &= \#\,\text{canceled } () \; + \; \#\,\text{canceled } \Big() \\ c'_{\alpha_2} &= \#\,\text{uncanceled } (+ \#\,\text{uncanceled } \Big(. \end{aligned} \tag{7}$$

3. Examples: Calculating crystal operators on PBW bases in general

For any two reduced expressions, we can understand the map $R_{\mathbf{i}}^{\mathbf{i}'} : \mathbf{Z}_{\ge 0}^N \to \mathbf{Z}_{\ge 0}^N$ sending $\boldsymbol{c}^{\mathbf{i}}(b)$ to $\boldsymbol{c}^{\mathbf{i}'}(b)$ by finding a way to move from $\mathbf{i}$ to $\mathbf{i}'$ by a sequence of braid moves, and composing the maps from §2.3. This gives a way to calculate any f_i.

Example 3.1. Let $\mathfrak{g}$ be of type A_4. Then $\mathbf{i} = (1,3,2,1,3,2,4,3,2,1)$ is a reduced expression, and the corresponding order on Φ^+ is

$$1 \prec 3 \prec 123 \prec 23 \prec 12 \prec 2 \prec 1234 \prec 234 \prec 34 \prec 4,$$

where 1 is identified with α_1, 1234 with $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, and so on. Consider

$$b = F_1^{(1)} F_3^{(1)} F_{123}^{(5)} F_{23}^{(3)} F_{12}^{(2)} F_2^{(3)} F_{1234}^{(4)} F_{234}^{(0)} F_{34}^{(1)} F_4^{(1)} \in \mathscr{B}_{\mathbf{i}}.$$

Calculating $f_1 b$ is easy: the exponent of F_1 just increases by 1. We now compute $f_2 b$:

$$\begin{array}{rclllllllllll}
b & = & F_1^{(1)} & F_3^{(1)} & F_{123}^{(5)} & F_{23}^{(3)} & F_{12}^{(2)} & F_2^{(3)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& \simeq & F_3^{(1)} & F_1^{(1)} & F_{123}^{(5)} & F_{23}^{(3)} & F_{12}^{(2)} & F_2^{(3)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& \simeq & F_3^{(1)} & F_{23}^{(7)} & F_{123}^{(1)} & F_1^{(5)} & F_{12}^{(2)} & F_2^{(3)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& \simeq & F_3^{(1)} & F_{23}^{(7)} & F_{123}^{(1)} & F_2^{(2)} & F_{12}^{(3)} & F_1^{(4)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& \simeq & F_3^{(1)} & F_{23}^{(7)} & F_2^{(2)} & F_{123}^{(1)} & F_{12}^{(3)} & F_1^{(4)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& \simeq & F_2^{(8)} & F_{23}^{(1)} & F_3^{(7)} & F_{123}^{(1)} & F_{12}^{(3)} & F_1^{(4)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)}, \\
f_2 b & \simeq & F_2^{(9)} & F_{23}^{(1)} & F_3^{(7)} & F_{123}^{(1)} & F_{12}^{(3)} & F_1^{(4)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)} \\
& & & & & & \vdots & & & & & \\
& \simeq & F_1^{(1)} & F_3^{(1)} & F_{123}^{(4)} & F_{23}^{(4)} & F_{12}^{(3)} & F_2^{(3)} & F_{1234}^{(4)} & F_{234}^{(0)} & F_{34}^{(1)} & F_4^{(1)}.
\end{array}$$

The first five steps perform braid moves and modify the PBW monomial according to the piecewise linear functions from Lemma 2.14. (To see this, recall that, by Lemma 2.13, two roots can be interchanged by a 2-term braid move exactly if they are perpendicular, and a 3-term braid move applies to consecutive roots β, β', β'' if and only if $\beta' = \beta + \beta''$.) Then F_2 is on the left, so to get $f_2 b$ just add one to its exponent. Then perform braid moves and the corresponding pieces of linear functions to get back to the original order.

Example 3.2. Let $\mathfrak{g}$ be of type D_4, and $\mathbf{i} = (1,2,3,4,2,1,2,3,4,2,3,4)$, where 2 is the trivalent node (see Figure (1)). The corresponding order on Φ^+ is

$$1 \prec 12 \prec 123 \prec 124 \prec 1234 \prec 12234 \prec 2 \prec 24 \prec 23 \prec 234 \prec 3 \prec 4,$$

where 1 is identified with α_1, 12 with $\alpha_1 + \alpha_2$, and so on. Consider

$$b = F_1^{(2)} F_{12}^{(1)} F_{123}^{(4)} F_{124}^{(2)} F_{1234}^{(1)} F_{12234}^{(3)} F_2^{(3)} F_{24}^{(1)} F_{23}^{(2)} F_{234}^{(1)} F_3^{(2)} F_4^{(0)} \in \mathscr{B}_{\mathbf{i}}.$$

The calculation of $f_4 b$, goes as follows:

$$\begin{array}{rclllllllllll}
b & = & F_1^{(2)} & F_{12}^{(1)} & F_{123}^{(4)} & F_{124}^{(2)} & F_{1234}^{(1)} & F_{12234}^{(3)} & F_2^{(3)} & F_{24}^{(1)} & F_{23}^{(2)} & F_{234}^{(1)} & F_3^{(2)} & F_4^{(0)} \\
& \simeq & F_1^{(2)} & F_{12}^{(1)} & F_{124}^{(2)} & F_{123}^{(4)} & F_{1234}^{(1)} & F_{12234}^{(3)} & F_2^{(3)} & F_{24}^{(1)} & F_{23}^{(2)} & F_{234}^{(1)} & F_4^{(0)} & F_3^{(2)} \\
& \simeq & F_1^{(2)} & F_{12}^{(1)} & F_{124}^{(2)} & F_{123}^{(4)} & F_{1234}^{(1)} & F_{12234}^{(3)} & F_2^{(3)} & F_{24}^{(1)} & F_4^{(1)} & F_{234}^{(0)} & F_{23}^{(3)} & F_3^{(2)} \\
& & & & & & \vdots & & & & & & & \\
& \simeq & F_4^{(2)} & F_1^{(2)} & F_{124}^{(1)} & F_{12}^{(2)} & F_{1234}^{(1)} & F_{123}^{(4)} & F_{12234}^{(3)} & F_{24}^{(1)} & F_2^{(3)} & F_{234}^{(0)} & F_{23}^{(3)} & F_3^{(2)}, \\
f_4 b & \simeq & F_4^{(3)} & F_1^{(2)} & F_{124}^{(1)} & F_{12}^{(2)} & F_{1234}^{(1)} & F_{123}^{(4)} & F_{12234}^{(3)} & F_{24}^{(1)} & F_2^{(3)} & F_{234}^{(0)} & F_{23}^{(3)} & F_3^{(2)} \\
& & & & & & \vdots & & & & & & & \\
& \simeq & F_1^{(2)} & F_{12}^{(1)} & F_{123}^{(3)} & F_{124}^{(2)} & F_{1234}^{(2)} & F_{12234}^{(3)} & F_2^{(3)} & F_{24}^{(1)} & F_{23}^{(2)} & F_{234}^{(1)} & F_3^{(2)} & F_4^{(0)}.
\end{array}$$

Example 3.3. In type C_3 where α_3 is the long root (see Figure (1)), set $\mathbf{i} = (1,2,3,2,1,2,3,2,3)$. The corresponding order on Φ^+ is

$$1 \prec 12 \prec 11223 \prec 123 \prec 1223 \prec 2 \prec 223 \prec 23 \prec 3,$$

where 1 is identified with α_1, 11223 is identified with $2\alpha_1 + 2\alpha_2 + \alpha_3$, and so on. Consider

$$b = F_1^{(4)} F_{12}^{(1)} F_{11223}^{(3)} F_{123}^{(2)} F_{1223}^{(0)} F_2^{(0)} F_{223}^{(5)} F_{23}^{(1)} F_3^{(2)} \in \mathscr{B}_{\mathbf{i}}.$$

Computing f_3:

$$\begin{array}{rccccccccc}
b & = & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_{1223}^{(0)} & F_2^{(0)} & F_{223}^{(5)} & F_{23}^{(1)} & F_3^{(2)} \\
 & \simeq & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_{1223}^{(0)} & F_3^{(6)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_3^{(6)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_3^{(7)} & F_{123}^{(1)} & F_{11223}^{(3)} & F_{12}^{(2)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_3^{(7)} & F_1^{(4)} & F_{123}^{(1)} & F_{11223}^{(3)} & F_{12}^{(2)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)}, \\
f_3 b & \simeq & F_3^{(8)} & F_1^{(4)} & F_{123}^{(1)} & F_{11223}^{(3)} & F_{12}^{(2)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_3^{(8)} & F_{123}^{(1)} & F_{11223}^{(3)} & F_{12}^{(2)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_3^{(7)} & F_{1223}^{(0)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_{1223}^{(0)} & F_3^{(7)} & F_{23}^{(0)} & F_{223}^{(2)} & F_2^{(7)} \\
 & \simeq & F_1^{(4)} & F_{12}^{(1)} & F_{11223}^{(3)} & F_{123}^{(2)} & F_{1223}^{(0)} & F_2^{(0)} & F_{223}^{(4)} & F_{23}^{(3)} & F_3^{(2)}.
\end{array}$$

Notice that the exponent of F_{23} increases by two.

4. Results

We now have a method to calculate crystal operators on any PBW monomial, but it can be a long process. However, for some words, some f_i can be calculated by a simple bracketing procedure. We discuss those words here. In fact, in most types there are a few very nice words where all the f_i are easy to calculate.

4.1. Simply braided words and bracketing rules

Fix $\mathbf{i} \in R(w_0)$. The elements of $B(\infty)$ are parameterized by their $\mathbf{i}$-Lusztig data, so we can think of $B(\infty)$ as a crystal structure on $\mathbf{Z}^N$. Specifically, $f_i(\boldsymbol{c}) = \boldsymbol{c}'$ if and only if there is some $b \in B(\infty)$ such that $\boldsymbol{c} = \boldsymbol{c}^{\mathbf{i}}(b)$ and $\boldsymbol{c}' = \boldsymbol{c}^{\mathbf{i}}(f_i b)$.

Definition 4.1. Fix a reduced expression $\mathbf{i}$ for w_0, and $i \in I$. We say that $\mathbf{i}$ is *simply braided for i* if one can perform a sequence of braid moves to $\mathbf{i}$ to get to a word $\mathbf{i}'$ with $i'_1 = i$, and each move in the sequence is either

- a 2-term braid move, or
- a braid move such that α_i is the rightmost of the roots affected.

We call $\mathbf{i}$ *simply braided* if it is simply braided for all $i \in I$.

Remark 4.2. Examples 3.2 and 3.3 both show computations in simply braided cases.

Fix $i \in I$ and $\mathbf{i} \in R(w_0)$ which is simply braided for i. Fix a sequence of braid moves as in Definition 4.1, and let $M_1, \ldots, M_k$ be the non-trivial braid moves. For each ℓ, let Δ_ℓ be the corresponding rank 2 root system.

Definition 4.3. Assume $\mathbf{i} \in R(w_0)$ is simply braided for i. Fix an $\mathbf{i}$-Lusztig datum $\boldsymbol{c}$. For each ℓ, let $\boldsymbol{c}^\ell$ be the rank 2 Lusztig datum defined by $c^\ell_{\alpha_i} = 0$, and for all other $\beta \in \Delta_\ell$, $c^\ell_\beta = c_\beta$. Let

$$R_{i;\ell}(\boldsymbol{c}) = \varepsilon_i(\boldsymbol{c}^\ell) \quad \text{and} \quad L_{i;\ell}(\boldsymbol{c}) = \langle \alpha_i^\vee, \mathrm{wt}(\boldsymbol{c}) \rangle + \varepsilon_i(\boldsymbol{c}^\ell) + \varepsilon_i^*(\boldsymbol{c}^\ell).$$

Set

$$S_i(\boldsymbol{c}) = \underbrace{)\cdots)}_{R_{i;k}} \underbrace{(\cdots(}_{L_{i;k}} \cdots \underbrace{)\cdots)}_{R_{i;2}} \underbrace{(\cdots(}_{L_{i;2}} \underbrace{)\cdots)}_{R_{i;1}} \underbrace{(\cdots(}_{L_{i;1}} \underbrace{)\cdots)}_{c_{\alpha_i}}.$$

Remark 4.4. The integer $L_{i;\ell}(\boldsymbol{c})$ is the number of times one may apply f_i to $\boldsymbol{c}^\ell$ before $c^\ell_{\alpha_i}$ becomes non-zero, and is sometimes called "jump"; see [LV11].

Theorem 4.5. *Fix $i \in I$ and assume $\mathbf{i} \in R(w_0)$ simply braided for i. Then, for any $\mathbf{i}$-Lusztig datum $\boldsymbol{c}$, $f_i(\boldsymbol{c})$ is*

- *the $\mathbf{i}$-Lusztig datum where $\boldsymbol{c}^k$ is changed to $f_i(\boldsymbol{c}^k)$ if the leftmost uncanceled "(" in $S_i(\boldsymbol{c})$ comes from a root in Δ_k, or*
- *the $\mathbf{i}$-Lusztig datum where c_{α_i} has increased by 1 if there is no uncanceled "(".*

Proof. Proceed by induction on the number $m_i(\mathbf{i})$ of braid moves required to change $\mathbf{i}$ to an i-adapted word, using only moves of the form allowed by Definition 4.1. The case where $i_1 = i$ is trivial. So, assume $\mathbf{i}'$ is related to $\mathbf{i}$ by an allowable braid move and that $m_i(\mathbf{i}') = m_i(\mathbf{i}) - 1$. If the braid move from $\mathbf{i}$ to $\mathbf{i}'$ is a 2-term braid move, then it does not change the order of the brackets in $S_i(\boldsymbol{c})$ and, by Lemmas 2.14 and 2.16 and Remark 2.17, it does not change any c_β, so does not change the values of $L_{i;j}$ and $R_{i;j}$. This is true both before and after applying f_i, so the result follows by induction.

If the braid move involves more then two terms, then by definition it must be the move M_1. First consider the case when the leftmost uncanceled "(" is in Δ_j for $j > 1$. Applying the move M_1 gives a new Lusztig datum, $\boldsymbol{c}'$, where c'_{α_i} is the number of uncanceled right brackets in the substring $)^{R_{i;1}(\boldsymbol{c})}(^{L_{i;1}(\boldsymbol{c})})^{c_{\alpha_i}}$. Doing this does not change the placement of the first uncanceled "(". By induction, f_i affects a root not in Δ_1, and acts as in the statement. Undoing the move M_1 gives the result.

It remains to consider the cases when either the leftmost uncanceled "(" is in Δ_1 or there is no uncanceled "(". Again start by applying M_1. The resulting Lusztig datum has no uncanceled "(", so by induction, applying f_i just adds one to c_{α_i}. Undoing M_1 gives the result of applying f_i. But this procedure has exactly applied f_i to the Lusztig data $\boldsymbol{c}^1$. See Figure 2 on the next page. $\square$

Remark 4.6. Theorem 4.5 implies that the sub-root systems $\Delta_1, \ldots, \Delta_k$ do not depend on the sequence of braid moves (provided it satisfies the conditions of Definition 4.1).

$$\begin{array}{cccccccccc}
F_1^{(1)} & F_{12}^{(2)} & F_{123}^{(1)} & F_{1234}^{(5)} & F_2^{(2)} & F_{23}^{(3)} & F_3^{(1)} & F_{234}^{(4)} & F_{34}^{(1)} & F_4^{(1)} \\
 &) & ((& &))) & ((&) & & & \\[1em]
F_1^{(1)} & F_{12}^{(2)} & F_{123}^{(1)} & F_{1234}^{(5)} & F_3^{(3)} & F_{23}^{(1)} & F_2^{(4)} & F_{234}^{(4)} & F_{34}^{(1)} & F_4^{(1)} \\
 &) & ((& &))) & & & & & \\[1em]
F_1^{(1)} & F_{12}^{(2)} & F_{123}^{(1)} & F_{1234}^{(5)} & F_3^{(4)} & F_{23}^{(1)} & F_2^{(4)} & F_{234}^{(4)} & F_{34}^{(1)} & F_4^{(1)} \\
 &) & ((& &)))) & & & & & \\[1em]
F_1^{(1)} & F_{12}^{(2)} & F_{123}^{(1)} & F_{1234}^{(5)} & F_2^{(1)} & F_{23}^{(4)} & F_3^{(1)} & F_{234}^{(4)} & F_{34}^{(1)} & F_4^{(1)} \\
 &) & ((& &)))) & (&) & & &
\end{array}$$

FIGURE 2. The inductive step in the proof of Theorem 4.5. Here we are applying f_3 in type A_3. At each step we have placed the string of brackets $S_3(\boldsymbol{c})$ below the monomial $F^{\boldsymbol{c}}$ (note, for example, that the brackets for the roots F_{12} and F_{123} need to be swapped). This is the final case, where the first uncanceled bracket is in $\boldsymbol{c}^1$. The first braid move is M_1, and involves the roots 2, 23, and 3. The new string of brackets after that move is obtained by deleting the brackets corresponding to these three roots, and replacing them with the number of uncanceled right brackets in the string we would use to apply f_3 in that rank 2 case. There were no uncanceled left brackets further to the left, so now there are no uncanceled left brackets at all, and, by induction, f_3 increases the exponent of F_3 by 1. Then we undo the move M_1. The result is the result of applying f_3 to the rank 2 monomial $F_2^{(2)}F_{23}^{(3)}F_3^{(1)}$.

Remark 4.7. The reduced expression in Example 3.1 is not simply braided, and hence the action of f_2 is more complicated: it affects three roots, and it follows from Theorem 4.5 that this cannot happen in simply braided simply-laced cases.

4.2. Existence of simply braided words

In this section, we show that any "good enumeration" of a Dynkin diagram, as defined in [Lit98], gives a simply braided word. These words are closely related to Littelmann's "nice decompositions." In particular, such words exist in all types except E_8 and F_4.

Definition 4.8. For $J \subseteq I$, let W_J denote the parabolic subgroup of W generated by J. Fix $i \in I$. Let τ^i be the minimal length representative of w_0 in $W_{I\backslash\{i\}}\backslash W$.

For each subset I' of I, let $\sigma^{I'}$ be the diagram automorphism of I' defined by $-w_0^{I'}$, where $w_0^{I'}$ is the longest element of the Weyl group for I'. Note that $\tau^i = (w_0^{I\backslash\{i\}})^{-1}w_0$, so $(\tau^i)^{-1}$ takes simple roots other than α_i to simple roots according to the map $(\sigma^I)^{-1}\sigma^{I\backslash\{i\}}$.

Definition 4.9. For any enumeration $i_1, \dots, i_n$ of I, let w be the minimal length representative of $w_0^{I\backslash\{i_1,\dots,i_{k-1}\}}$ in $W_{I\backslash\{i_1,\dots,i_k\}}\backslash W_{I\backslash\{i_1,\dots,i_{k-1}\}}$ and define

$$\tau^{(k)} = (\sigma^I)^{-1}\sigma^{I\backslash\{i_1,\dots,i_{k-1}\}}w.$$

Lemma 4.10. *The reduced expression of w_0 corresponding to the convex order from Definition 2.7 factors as $\tau^{(1)}\tau^{(2)}\cdots\tau^{(n)}$.*

Example 4.11. Consider type D_5, where 3 is the trivalent node, and the enumeration $i_1 = 4$, $i_2=2$, $i_3=3$, $i_4=1$, and $i_5=5$. The corresponding order on roots is

$$4 \prec 34 \prec 345 \prec 234 \prec 2345 \prec 1234 \prec 12345 \prec 23345 \prec 123345 \prec 1223345 \\ \prec 2 \prec 12 \prec 23 \prec 235 \prec 123 \prec 1235 \prec 3 \prec 35 \prec 1 \prec 5,$$

which corresponds to the reduced expression

$$s_4s_3s_5s_2s_3s_1s_2s_4s_3s_5s_3s_4s_2s_1s_3s_2s_4s_3s_1s_4.$$

Then

$$s_4s_3s_5s_2s_3s_1s_2s_4s_3s_5 = \tau^{(1)},$$
$$s_3s_4s_2s_1s_3s_2 = \sigma^{-1}\sigma^{\{1,2,3,5\}}s_2s_1s_3s_5s_2s_3 = \tau^{(2)},$$
$$s_4s_3 = \sigma^{-1}\sigma^{\{1,3,5\}}s_3s_5 = \tau^{(3)}, \ \ s_1 = \tau^{(4)}, \ \ \text{and} \ \ s_4 = \tau^{(5)}.$$

Here σ is the diagram automorphism swapping 4 and 5, $\sigma^{\{1,2,3,5\}}$ reverses the order of that A_4 Dynkin diagram, $\sigma^{\{1,3,5\}}$ swaps 3 and 5, and $\sigma^{\{1,5\}}$ and $\sigma^{\{5\}}$ are trivial. Notice that all pairs of roots of the form $\beta, \beta+\alpha_5$ are adjacent, as in Lemma 4.12 below.

Proof of Lemma 4.10. The convex order from Definition 2.7 is of the form

$$\beta_{1,1} \prec \beta_{1,2} \prec \cdots \prec \beta_{1,k_1} \prec \beta_{2,1} \prec \cdots \prec \beta_{1,k_2} \prec \cdots \prec \beta_{(n-1),k_{n-1}} \prec \beta_{n,1},$$

where

- $\{\beta_{1,1}, \beta_{1,2}, \dots, \beta_{1,k_1}\}$ is exactly those roots $\beta = \sum_{j\in I} c_j\alpha_j$ with $c_{i_1} > 0$,
- $\{\beta_{2,1}, \dots, \beta_{2,k_2}\}$ is those roots $\beta = \sum_{j\in I} c_j\alpha_j$ with $c_{i_1} = 0$ and $c_{i_2} > 0$, and so on.

Consider the corresponding reduced expression, factored as $w^{(1)}w^{(2)}\cdots w^{(n)}$, where $w^{(j)}$ corresponds to the roots $\beta_{j,1}, \dots, \beta_{j,k_j}$. Then $\{\beta_{1,1}, \beta_{1,2}, \dots, \beta_{1,k_1}\}$ is the subset of Φ^+ sent to negative roots by $(w^{(1)})^{-1}$, so in particular $(w^{(1)})^{-1}$ does not send any simple root other than α_{i_1} to a negative root. This implies that every reduced expression of $(w^{(1)})^{-1}$ has s_{i_1} on the right, so $(w^{(1)})^{-1}$ is a minimal length coset representative in $W/W_{I\setminus\{i_1\}}$. Equivalently, $w^{(1)}$ is a minimal length coset representative in $W_{I\setminus\{i_1\}}\backslash W$.

For any $w\in W$ longer than $w^{(1)}$, w^{-1} must take some root not in $\{\beta_{1,1}, \dots, \beta_{1,k_1}\}$ to a negative root, so, since the coefficient of α_{i_1} in that root is 0, it must take some other simple root to a negative root. This implies that w is not a minimal length representative of its coset in $W_{I\setminus\{i_1\}}\backslash W$. Thus $w^{(1)}$ is in fact equal to $\tau^{(1)}$.

Let $a\colon I\setminus\{i_1\} \to I$ denote the map $(\sigma^I)^{-1}\sigma^{I\setminus\{i_1\}}$. Since $\tau^{(1)}$ performs the map a^{-1} on roots, $w^{(2)}\cdots w^{(n)}$ must be the reduced expression for the root system $a(I\setminus\{i_1\})$ corresponding to the convex order

$$a\beta_{2,1} \prec \cdots \prec a\beta_{1,k_2} \prec \cdots \prec a\beta_{(n-1),k_{n-1}} \prec a\beta_{n,1}.$$

By induction, this factors as $\eta^{(2)}\cdots\eta^{(n)}$, where

$$\eta^{(k)} = a(\sigma^{I\setminus\{i_1\}})^{-1}\sigma^{I\setminus\{i_1,\dots,i_{k-1}\}}w$$

and w is the representative of $w_0^{I\setminus\{i_1,\dots,i_{k-1}\}}$ in $W_{I\setminus\{i_1,\dots,i_k\}}\backslash W_{I\setminus\{i_1,\dots,i_{k-1}\}}$ of minimal length. The result follows by substituting $a = (\sigma^I)^{-1}\sigma^{I\setminus\{i_1\}}$ and simplifying. □

Lemma 4.12. *For any enumeration $i_1, \ldots, i_n$ of I, the convex order from Definition* 2.7 *and Lemma* 2.9 *has the property that, for any rank two sub-root system $\Delta_\circ$ with simple roots β and α_{i_n}, all the roots except possibly α_{i_n} are adjacent.*

Proof. This is immediate from Definition 2.7, since all roots in span β, α_n other than β are equal in all the comparisons but the very last, so they are adjacent. □

Definition 4.13. Following Littelmann [Lit98], $i \in I$ is called *braidless*, if for any $\tau \in W$, the following holds: if $\alpha, \gamma \in \Phi$ are such that $\langle \alpha^\vee, \tau\omega_i \rangle > 0$ and $\langle \gamma^\vee, \tau\omega_i \rangle > 0$, then $\langle \alpha^\vee, \gamma \rangle = 0$. Here ω_i is the corresponding fundamental weight.

Definition 4.14. An enumeration $i_1, \ldots, i_n$ of I is called a *good enumeration* if each i_k is braidless for the root system generated by $\{\alpha_{i_k}, \ldots, \alpha_{i_n}\}$.

By [Lit98, Lem. 3.1], a node of a Dynkin diagram is braidless exactly if the corresponding fundamental weight is either minuscule or co-minuscule (meaning it is minuscule for the Dynkin diagram with arrows reversed), or the Dynkin diagram is rank two. Using the classification of minuscule weights in Table 1 we see:

Proposition 4.15. *The following is a complete list of good enumerations of $I = \{i_1, \ldots, i_n\}$ depending on the underlying type.*

(i) *If $\mathfrak{g}$ is of type A_n, then any order of I is a good enumeration.*
(ii) *If $\mathfrak{g}$ if of type B_n or C_n, then order I as $1 < 2 < 3 \cdots < k < n < \cdots$ for some $k \geq 0$, where everything after the n is arbitrary. The $k = 0$ case means n is first in the order.*
(iii) *If $\mathfrak{g}$ is of type D_n, then order I as $1 < 2 < \cdots < k < x < \cdots$ for some $k \geq 0$, where $x = n-1$ or n, and everything after x is arbitrary.*
(iv) *If $\mathfrak{g}$ is of type E_6, then set i_1 to be one of the two vertices farthest from the trivalent vertex in the Dynkin diagram, and then follow type D_5 rules.*
(v) *If $\mathfrak{g}$ is of type E_7, then set i_1 to be the vertex farthest from the trivalent node, and then follow type E_6 rules.* □

A_n	$1, \ldots, n$
B_n	n
C_n	1
D_n	$1, n-1, n$
E_6	$1, 6$
E_7	7
$\mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$	none

TABLE 1. The indices of minuscule fundamental weights, using the indexing of fundamental weights from [Bou02]. This table is from [Bou05, p. 132].

Braidless nodes are useful because of the following property.

Lemma 4.16 ([Lit98, Lem. 3.2])**.** *If i is braidless for I, then τ^i has a unique reduced expression, up to 2-term braid moves.* □

Lemma 4.17. *For any good enumeration of I, the corresponding convex order from Lemma 2.7 is simply braided.*

Proof. Fix i, and let k be such that $i_k = i$. Recall from Lemma 4.10 that the corresponding reduced expression factors as $\tau^{(1)}\tau^{(2)}\cdots\tau^{(n)}$. Let X be the set of roots corresponding to $\tau^{(1)}\cdots\tau^{(k-1)}$. Consider two convex orders defined using Lemma 2.9:

- $\prec$ defined using the order $i_1, i_2 \dots, i_n$, and
- $\prec'$ defined by $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n, i_k$.

Consider the hybrid order $\prec''$ where X is ordered according to $\prec'$, and the rest of the roots are ordered according to $\prec$. By Lemma 2.6 this remains convex. Then:

(i) $\prec$ and $\prec''$ both factor as a product of $\tau^{(j)}$ in the same order, so, since we are working with a good enumeration, they differ by 2-term braid moves;
(ii) the first root coming from $\tau^{(k)}$ in $\prec$ is α_i;
(iii) by Lemma 4.12, all roots β that are $\prec'' \alpha_i$ which satisfy $(\beta|\alpha_i) < 0$ have the property that the roots in the rank 2 root system including β and α_i are adjacent according to $\prec''$. From this it follows that $\prec''$ is simply braided for i.

Together these certainly imply that $\prec$ is simply braided for i. See Example 4.18. □

Example 4.18. Consider type D_4 and the good enumeration $i_1 = 1$, $i_2 = 3$, $i_3 = 2$, and $i_4 = 4$. We illustrate the proof that the convex order from Lemma 2.7 is simply braided for node 3. The convex orders considered are

$$1 \prec 12 \prec 124 \prec 123 \prec 1234 \prec 12234 \prec 3 \prec 23 \prec 234 \prec 2 \prec 24 \prec 4$$
$$1 \prec' 12 \prec' 123 \prec' 124 \prec' 1234 \prec' 12234 \prec' 2 \prec' 23 \prec' 24 \prec' 234 \prec' 4 \prec' 3,$$

and the hybrid order

$$1 \prec'' 12 \prec'' 123 \prec'' 124 \prec'' 1234 \prec'' 12234 \prec'' 3 \prec'' 23 \prec'' 234 \prec'' 2 \prec'' 24 \prec'' 4$$

which follows $\prec'$ up to 12234, then follows $\prec$. The order $\prec'$ is simply braided for 3 by Lemma 4.12, so $\prec''$ is simply braided for 3 as well because it is the same order as $\prec'$ to the left of the root 3. Both $\prec$ and $\prec''$ factor as a product of $\tau^{(j)}$ in the same order, so, since we are using a good enumeration, they differ only by 2-term braid moves, and hence $\prec$ is simply braided for 3 by definition. This last step fails for other enumerations.

4.3. Bracketing rules in rank 2

We will use the notation $f_i(\boldsymbol{c})_\beta$ to mean the β-component of $f_i(\boldsymbol{c})$, where $\boldsymbol{c}$ is an $\mathbf{i}$-Lusztig datum for some $\mathbf{i} \in R(w_0)$ and β is a positive root.

Lemma 4.19. *Let $\mathfrak{g} = \mathfrak{sl}_3$ and set $\mathbf{i} = (1,2,1)$. For any $\mathbf{i}$-Lusztig datum $\boldsymbol{c} = (c_{\alpha_1}, c_{\alpha_1+\alpha_2}, c_{\alpha_2})$, $f_2(\boldsymbol{c})$ can be calculated as follows. Make the string of brackets*

$$S_2(\boldsymbol{c}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}} \;\; \underbrace{(\cdots(}_{c_{\alpha_1}} \;\; \underbrace{)\cdots)}_{c_{\alpha_2}}.$$

Then $\boldsymbol{c}$ and $f_2(\boldsymbol{c})$ differ as follows.

- *If there is an uncanceled "(", then* $f_2(\boldsymbol{c})_{\alpha_1+\alpha_2} = c_{\alpha_1+\alpha_2} + 1$ *and* $f_2(\boldsymbol{c})_{\alpha_1} = c_{\alpha_1} - 1$.
- *If there is no uncanceled "(", then* $f_2(\boldsymbol{c})_{\alpha_2} = c_{\alpha_2} + 1$.

Proof. Let $\boldsymbol{c}' = (c'_{\alpha_2}, c'_{\alpha_1+2\alpha_2}, c'_{\alpha_1}) = R^{212}_{121}(\boldsymbol{c})$, where $R^{\mathbf{i}'}_{\mathbf{i}}$ is the transition map from Lemma 2.14. If there is an uncanceled left bracket, then $c_{\alpha_1} > c_{\alpha_2}$. Hence $\boldsymbol{c}' = (c_{\alpha_1+\alpha_2}, c_{\alpha_2}, c_{\alpha_1} + c_{\alpha_1+\alpha_2} - c_{\alpha_2})$. Using Proposition 2.12,

$$f_2(\boldsymbol{c}') = (c_{\alpha_1+\alpha_2} + 1, c_{\alpha_2}, c_{\alpha_1} + c_{\alpha_1+\alpha_2} - c_{\alpha_2}).$$

Applying the inverse to R^{212}_{121} gives $f_2(\boldsymbol{c}) = (c_{\alpha_1} - 1, c_{\alpha_1+\alpha_2} + 1, c_{\alpha_2})$, as required.

If there is no uncanceled left bracket, then $c_{\alpha_1} \leq c_{\alpha_2}$, so $\boldsymbol{c}' = (c_{\alpha_1+\alpha_2} + c_{\alpha_2} - c_{\alpha_1}, c_{\alpha_1}, c_{\alpha_1+\alpha_2})$. Using Proposition 2.12,

$$f_2(\boldsymbol{c}') = (c_{\alpha_1+\alpha_2} + c_{\alpha_2} - c_{\alpha_1} + 1, c_{\alpha_1}, c_{\alpha_1+\alpha_2}).$$

Applying the inverse to R^{212}_{121} gives $f_2(\boldsymbol{c}) = (c_{\alpha_1}, c_{\alpha_1+\alpha_2}, c_{\alpha_2} + 1)$, as required. $\square$

Lemma 4.20. *Let* $\mathfrak{g}$ *be of type* B_2 *where* α_2 *is the short root and set* $\mathbf{i} = (1, 2, 1, 2)$. *For any* $\mathbf{i}$*-Lusztig datum* $\boldsymbol{c} = (c_{\alpha_1}, c_{\alpha_1+\alpha_2}, c_{\alpha_1+2\alpha_2}, c_{\alpha_2})$, $f_2(\boldsymbol{c})$ *can be calculated as follows. Make the string of brackets*

$$S_2(\boldsymbol{c}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}}\ \underbrace{(\cdots(}_{2c_{\alpha_1}}\ \underbrace{)\cdots)}_{2c_{\alpha_1+2\alpha_2}}\ \underbrace{(\cdots(}_{c_{\alpha_1+\alpha_2}}\ \underbrace{)\cdots)}_{c_{\alpha_2}}.$$

Then $\boldsymbol{c}$ *and* $f_2(\boldsymbol{c})$ *differ as follows.*

(i) *If the leftmost uncanceled "(" corresponds to* α_1*, then* $f_2(\boldsymbol{c})_{\alpha_1} = c_{\alpha_1} - 1$ *and* $f_2(\boldsymbol{c})_{\alpha_1+\alpha_2} = c_{\alpha_1+\alpha_2} + 1$.
(ii) *If the leftmost uncanceled "(" corresponds to* $\alpha_1 + \alpha_2$*, then* $f_2(\boldsymbol{c})_{\alpha_1+\alpha_2} = c_{\alpha_1+\alpha_2} - 1$ *and* $f_2(\boldsymbol{c})_{\alpha_1+2\alpha_2} = c_{\alpha_1+2\alpha_2} + 1$.
(iii) *If there is no uncanceled "(", then* $f_2(\boldsymbol{c})_{\alpha_2} = c_{\alpha_2} + 1$.

Proof. Let $\overline{\boldsymbol{c}}$ denote the Lusztig data proposed for $f_2(\boldsymbol{c})$. Let $\boldsymbol{c}'$ be the Lusztig data obtained by performing a braid move on $\boldsymbol{c}$, and $\overline{\boldsymbol{c}}'$ the data obtained by performing a braid move on $\overline{\boldsymbol{c}}$. It suffices to show that $\boldsymbol{c}'$ and $\overline{\boldsymbol{c}}'$ differ only by $\overline{c}'_{\alpha_2} = c'_{\alpha_2} + 1$. We compare the strings of brackets for $T(\boldsymbol{c})$ and $T(\overline{\boldsymbol{c}})$ from Lemma 2.16. The cases are broken down according to the itemization of the statement.

In case (i),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots))}_{c_{\alpha_1+\alpha_2}+1}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1}-1}\ \underbrace{\Big)\cdots\Big)}_{c_{\alpha_1+2\alpha_2}}\ \underbrace{((\cdots(}_{c_{\alpha_1+\alpha_2}+1}\ \underbrace{)\cdots)}_{c_{\alpha_2}}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1+2\alpha_2}}.$$

Here one bracket corresponding to α_1 has been removed and one of each type of bracket corresponding to $\alpha_1 + \alpha_2$ has been added. The condition that the leftmost uncanceled "(" in $S_2(\boldsymbol{c})$ corresponds to α_1 implies one of the following.

- There is a pair $\Big()$ in $T(\boldsymbol{c})$. Then $T(\overline{\boldsymbol{c}})$ has one less $\Big()$, one more $()$, and one more $)$.
- The leftmost uncanceled left bracket in $T(\boldsymbol{c})$ comes from α_1 and there is no $\Big()$. Then $T(\overline{\boldsymbol{c}})$ has one less uncanceled $\Big($ and one more uncanceled $($ and $)$. The new $($ is uncanceled since otherwise there would have been a $\Big()$.

In case (ii),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}-1}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1}}\ \underbrace{\Big)\cdots\Big)\Big)}_{c_{\alpha_1+2\alpha_2}+1}\ \underbrace{(\cdots(}_{c_{\alpha_1+\alpha_2}-1}\ \underbrace{)\cdots)}_{c_{\alpha_2}}\ \underbrace{\Big(\Big(\cdots\Big(}_{c_{\alpha_1+2\alpha_2}+1}.$$

Comparing to $T(\boldsymbol{c})$, there is one less uncanceled (, one less uncanceled), one more uncanceled $\Big)$, and one more uncanceled $\Big($.

In case (iii),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1}}\ \underbrace{\Big)\cdots\Big)}_{c_{\alpha_1+2\alpha_2}}\ \underbrace{(\cdots(}_{c_{\alpha_1+\alpha_2}}\ \underbrace{)\cdots))}_{c_{\alpha_2}+1}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_1+2\alpha_2}}.$$

There is one new uncanceled).

In each case, it is easy to see using Lemma 2.16 that $\overline{\boldsymbol{c}}'$ is as desired. □

Lemma 4.21. *Let $\mathfrak{g}$ be of type C_2 where α_2 is the long root and set $\mathbf{i} = (1,2,1,2)$. For any $\mathbf{i}$-Lusztig datum $\boldsymbol{c} = (c_{\alpha_1}, c_{2\alpha_1+\alpha_2}, c_{\alpha_1+\alpha_2}, c_{\alpha_2})$, $f_2(\boldsymbol{c})$ can be calculated as follows. Make the string of brackets*

$$S_2(\boldsymbol{c}) = \underbrace{)\cdots)}_{c_{2\alpha_1+\alpha_2}}\ \underbrace{(\cdots(}_{c_{\alpha_1}}\ \underbrace{)\cdots)}_{c_{\alpha_1+\alpha_2}}\ \underbrace{(\cdots(}_{c_{2\alpha_1+\alpha_2}}\ \underbrace{)\cdots)}_{c_{\alpha_2}}.$$

Then $\boldsymbol{c}$ and $f_2(\boldsymbol{c})$ differ as follows.

(i) *If the leftmost uncanceled "(" corresponds to α_1 and $c_{\alpha_1+\alpha_2} = c_{\alpha_1}-1$, then $f_2(\boldsymbol{c})_{\alpha_1} = c_{\alpha_1} - 1$ and $f_2(\boldsymbol{c})_{\alpha_1+\alpha_2} = c_{\alpha_1+\alpha_2} + 1$.*

(ii) *If the leftmost uncanceled "(" corresponds to α_1 and $c_{\alpha_1+\alpha_2} < c_{\alpha_1}-1$, then $f_2(\boldsymbol{c})_{\alpha_1} = c_{\alpha_1} - 2$ and $f_2(\boldsymbol{c})_{2\alpha_1+\alpha_2} = c_{2\alpha_1+\alpha_2} + 1$.*

(iii) *If the leftmost uncanceled "(" corresponds to $2\alpha_1+\alpha_2$, then $f_2(\boldsymbol{c})_{2\alpha_1+\alpha_2} = c_{\alpha_1+\alpha_2} - 1$ and $f_2(\boldsymbol{c})_{\alpha_1+\alpha_2} = c_{\alpha_1+2\alpha_2} + 2$.*

(iv) *If there is no uncanceled "(", then $f_2(\boldsymbol{c})_{\alpha_2} = c_{\alpha_2} + 1$.*

Proof. Let $\overline{\boldsymbol{c}}$ denote the Lusztig data proposed for $f_2(\boldsymbol{c})$, and $\boldsymbol{c}'$, $\overline{\boldsymbol{c}}'$ the data obtained by performing a braid move on $\boldsymbol{c}$ and $\overline{\boldsymbol{c}}$ respectively. It suffices to show that $\boldsymbol{c}'$ and $\overline{\boldsymbol{c}}'$ differ only by $\overline{c}'_{\alpha_2} = c'_{\alpha_2}+1$. Consider the string of brackets for $T(\boldsymbol{c})$ from Lemma 2.16, where, since here α_1 is the long root, the roles of the two roots are interchanged. The roots in $T(\boldsymbol{c})$ come in the reverse order of those in $S_2(\boldsymbol{c})$, so the conditions on the left brackets in $S_2(\boldsymbol{c})$ give information about right brackets in $T(\boldsymbol{c})$.

In case (i),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots))}_{c_{\alpha_2+\alpha_1}+1}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2}}\ \underbrace{\Big)\cdots\Big)}_{c_{\alpha_2+2\alpha_1}}\ \underbrace{((\cdots(}_{c_{\alpha_2+\alpha_1}+1}\ \underbrace{)\cdots)}_{c_{\alpha_1}-1}\ \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2+2\alpha_1}}.$$

The conditions imply that all of the brackets added and removed are uncanceled, so in total $T(\overline{\boldsymbol{c}})$ has one more (than $T(\boldsymbol{c})$.

In case (ii),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots)}_{c_{\alpha_2+\alpha_1}} \; \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2}} \; \underbrace{\Big)\cdots\Big)\Big)}_{c_{\alpha_2+2\alpha_1}+1} \; \underbrace{(\cdots(}_{c_{\alpha_2+\alpha_1}} \; \underbrace{)\cdots)}_{c_{\alpha_1}-2} \; \underbrace{\Big(\Big(\cdots\Big(}_{c_{\alpha_2+2\alpha_1}+1} .$$

This breaks into two cases:

- $c_{\alpha_2+2\alpha_1} \geq c_{\alpha_2}$, so there are no uncanceled $\Big)$ nor $\Big($)). Then there are two less), one more $\Big)$, and one more $\Big($.
- $c_{\alpha_2+2\alpha_1} < c_{\alpha_2}$. Since there is an uncanceled right bracket corresponding to α_1, $T(\boldsymbol{c})$ has a $\Big($)). So there are one more $\Big(\Big)$ and $\Big($ and one less $\Big($)).

In case (iii),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots)))}_{c_{\alpha_2+\alpha_1}+2} \; \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2}} \; \underbrace{\Big)\cdots\Big)}_{c_{\alpha_2+2\alpha_1}-1} \; \underbrace{(((\cdots(}_{c_{\alpha_2+\alpha_1}+2} \; \underbrace{)\cdots)}_{c_{\alpha_1}} \; \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2+2\alpha_1}-1} .$$

There are two more uncanceled) and (, and one less uncanceled $\Big)$ and $\Big($.

For case (iv),

$$T(\overline{\boldsymbol{c}}) = \underbrace{)\cdots)}_{c_{\alpha_2+\alpha_1}} \; \underbrace{\Big(\Big(\cdots\Big(}_{c_{\alpha_2}+1} \; \underbrace{\Big)\cdots\Big)}_{c_{\alpha_2+2\alpha_1}} \; \underbrace{(\cdots(}_{c_{\alpha_2+\alpha_1}} \; \underbrace{)\cdots)}_{c_{\alpha_1}} \; \underbrace{\Big(\cdots\Big(}_{c_{\alpha_2+2\alpha_1}} .$$

The string $T(\boldsymbol{c})$ has no uncanceled) and no $\Big($)), so the new $\Big($ is uncanceled.

In each case, it is easy to see using Lemma 2.16 that $\overline{\boldsymbol{c}}'$ is as desired. □

Remark 4.22. The reader may wonder why we use only small brackets in the statement of Lemmas 4.20 and 4.21, as opposed to including the big brackets from Lemma 2.16. Here the big brackets do seem natural, but later on when we consider multiple braid moves it seems best to convert to one type of bracket.

4.4. General bracketing rules

Theorem 4.5, along with §4.3, gives a realization of $B(\infty)$ by bracketing rules for any simply braided reduced expression $\mathbf{i}$. Fix $i \in I$ and let $M_1, \ldots, M_k$ be the nontrivial braid moves required to move α_i to the left of the convex order. Let $\Delta_1, \ldots, \Delta_k$ be the corresponding rank two sub-root-systems and, for each $1 \leq \ell \leq k$, let $\boldsymbol{c}^\ell$ be the rank 2 Lusztig datum defined by $c^\ell_{\alpha_i} = 0$, and $c^\ell_\beta = c_\beta$ for all other $\beta \in \Delta_\ell$. Let S_i^ℓ be the string of brackets corresponding to $\boldsymbol{c}^\ell$ as in §4.3. Let

$$S_i(\boldsymbol{c}) = S_i^k \cdots S_i^2 S_i^1 \;)^{c_{\alpha_i}}. \tag{8}$$

Then $S_i(\boldsymbol{c})$ can be used to compute f_i.

5. Examples: Calculating crystal operators using bracketing rules

We now discuss a specific braidless reduced expression in each of the classical types, and the resulting crystal structures. All of these reduced expressions come from good enumerations, as in [Lit98]. There are actually many other good enumerations, and hence other braidless reduced expressions. It could be interesting to understand the combinatorics coming from these other enumerations.

In each of the following subsections we fix a particular reduced expression and use it to identify Lusztig data with Kostant partitions.

Remark 5.1. Below we often denote roots by stacks of numbers. For instance,

$$\begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 2 & \\ & 1 & \end{smallmatrix}$$

corresponds to the root $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_3$ in type D_4. The ordering of these stacks is chosen for the following reason: a root corresponds to a left bracket in S_i exactly if one can place an i at the top of the stack and still have either a root or a sum of roots. It corresponds to a right bracket in S_i exactly if one can remove an i from the top of the stack and still have either a root or a sum of roots.

5.1. Type A_n

The set of positive roots is $\{\alpha_{i,j} : 1 \le i \le j \le n\}$, where $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. The word $\mathbf{i}^{\mathsf{A}}$ corresponding to the reduced expression

$$w_0 = (s_1 s_2 \cdots s_n)(s_1 s_2 \cdots s_{n-1}) \cdots (s_1 s_2) s_1$$

is simply braided by Lemma 4.17. The corresponding order on positive roots $\alpha_{i,j} \prec \alpha_{i',j'}$ if and only if $i < i'$ or $i = i'$ and $j < j'$. Then, given $i \in I$ and an $\mathbf{i}^{\mathsf{A}}$-Lusztig datum $\boldsymbol{c} = (c_\beta)_{\beta \in \Phi^+}$, the string of brackets $S_i(\boldsymbol{c})$ is

$$\underbrace{)\cdots)}_{c_{\alpha_{1,i}}}\ \underbrace{(\cdots(}_{c_{\alpha_{1,i-1}}}\ \underbrace{)\cdots)}_{c_{\alpha_{2,i}}}\ \underbrace{(\cdots(}_{c_{\alpha_{2,i-1}}}\ \ \cdots\ \ \underbrace{)\cdots)}_{c_{\alpha_{i-i,i}}}\ \underbrace{(\cdots(}_{c_{\alpha_{i-i,i-1}}}\ \underbrace{)\cdots)}_{c_{\alpha_{i,i}}}.$$

Example 5.2. Consider type A_3 with $\mathbf{i} = (1,2,3,1,2,1)$ and $i = 2$. The corresponding order on positive roots is $1 \prec \substack{2\\1} \prec \substack{3\\2\\1} \prec 2 \prec \substack{3\\2} \prec 3$. If $b \in \mathscr{B}_{\mathbf{i}}$ is such that $c^{\mathbf{i}}(b) = (2,3,1,3,3,2)$, then the corresponding Kostant partition is

$$1\ 1\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ 2\ 2\ 2\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\\2}\ 3\ 3.$$

Placing the parts/roots in the order prescribed by Definition 4.3, we get

$$\begin{array}{rcccccccc} & \substack{2\\1} & \substack{2\\1} & \substack{2\\1} & 1 & 1 & 2 & 2 & 2 \\ S_2 : &) &) &) & \cancel{(} & \cancel{(} & \cancel{)} & \cancel{)} &) \ . \end{array}$$

There are no uncanceled left brackets, so applying f_2 yields

$$1\ 1\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ 2\ 2\ 2\ 2\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\\2}\ 3\ 3.$$

5.2. Type D_n

By Lemma 4.17 the word $\mathbf{i}^{\mathsf{D}}$ associated to the reduced expression

$$w_0 = (s_1 s_2 \cdots s_{n-1} s_n s_{n-2} \cdots s_2 s_1) \cdots (s_{n-2} s_{n-1} s_n s_{n-2}) s_{n-1} s_n$$

is simply braided.

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \leq i < k \leq n$
$\beta_{i,k} = \epsilon_i - \epsilon_{k+1},$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \epsilon_i + \epsilon_k,$	$1 \leq i < k \leq n$

TABLE 2. Positive roots of type D_n, expressed both as a linear combination of simple roots and in the canonical realization following [Bou02].

Using notation from Table 2, the order of the positive roots corresponding to the subword $(i, i+1, \ldots, n, n-2, \ldots, i)$ of $\mathbf{i}^{\mathsf{D}}$, for $1 \leq i \leq n-2$, is

$$\begin{cases} \beta_{i,i} \prec \cdots \prec \beta_{i,n-2} \prec \beta_{i,n-1} \prec \gamma_{i,n} \prec \gamma_{i,n-1} \prec \cdots \prec \gamma_{i,i+1} & \text{if } i \equiv 1 \bmod 2, \\ \beta_{i,i} \prec \cdots \prec \beta_{i,n-2} \prec \gamma_{i,n} \prec \beta_{i,n-1} \prec \gamma_{i,n-1} \prec \cdots \prec \gamma_{i,i+1} & \text{if } i \equiv 0 \bmod 2. \end{cases}$$

The ordering on the roots corresponding to the suffix $(n-1, n)$ of $\mathbf{i}^{\mathsf{D}}$ is

$$\begin{cases} \beta_{n-1,n-1} \prec \gamma_{n-1,n} & \text{if } n \equiv 0 \bmod 2, \\ \gamma_{n-1,n} \prec \beta_{n-1,n-1} & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

It follows that, for a Kostant partition $\boldsymbol{c} = (c_\beta)_{\beta \in \Phi^+}$, the string of brackets $S_i^{\mathbf{i}^{\mathsf{D}}}(\boldsymbol{c})$ is

$$\underbrace{)\cdots)}_{c_{\beta_{1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}}\ \underbrace{)\cdots)}_{c_{\beta_{i,i}}},$$

if $i \neq n$, and is

$$\underbrace{)\cdots)}_{c_{\gamma_{1,n}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,n-2}}}\ \underbrace{)\cdots)}_{c_{\gamma_{1,n-1}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,n-1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\gamma_{n-2,n}}}\ \underbrace{(\cdots(}_{c_{\beta_{n-2,n-2}}}\ \underbrace{)\cdots)}_{c_{\gamma_{n-2,n-1}}}\ \underbrace{(\cdots(}_{c_{\beta_{n-2,n-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{n-1,n}}},$$

if $i = n$.

Example 5.3. Consider the setup from Example 3.2. The Kostant partition corresponding to that datum is

$$1\ 1\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{4\\2\\1}\ \substack{4\\2\\1}\ \substack{3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ 2\ 2\ 2\ \substack{4\\2}\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\,4\\2}\ 3\ 3.$$

Arranging the parts/roots in the order prescribed by Definition 4.3, we get

$$\begin{array}{rcccccccccccccccc} & \substack{4\\2\\1} & \substack{4\\2\\1} & \substack{2\\1} & \substack{3\,4\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{4\\2} & 2 & 2 & 2 & \substack{3\,4\\2} & \substack{3\\2} & \substack{3\\2} & \\ S_4 : &) &) & \cancel{(} & \cancel{)} & (& (& (& \cancel{(} & \cancel{)} & (& (& \cancel{(} & \cancel{)} & (& (& . \end{array}$$

Hence applying f_4 gives

$$1\ 1\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{4\\2\\1}\ \substack{4\\2\\1}\ \substack{3\,4\\2\\1}\ \substack{3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ \substack{2\\3\,4\\2\\1}\ 2\ 2\ 2\ \substack{4\\2}\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\,4\\2}\ 3\ 3.$$

5.3. Types B_n and C_n

These have the same Weyl group. By Lemma 4.17, the word $\mathbf{i}^{\mathsf{BC}}$ associated to the reduced expression

$$w_0 = (s_1 s_2 \dots s_{n-1} s_n s_{n-1} \cdots s_2 s_1)(s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2) \cdots (s_{n-1} s_n s_{n-1}) s_n$$

is simply braided.

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k \le n$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_n,$	$1 \le i < k \le n$
$\beta_{i,k} = \epsilon_i - \epsilon_{k+1},$	$1 \le i \le k \le n-1$
$\beta_{i,n} = \epsilon_i,$	$1 \le i \le n$
$\gamma_{i,k} = \epsilon_i + \epsilon_k,$	$1 \le i < k \le n$

TABLE 3. Positive roots of type B_n, expressed both as a linear combination of simple roots and in the canonical realization following [Bou02].

The positive roots for type B_n are listed in Table 3, and the convex order on Φ^+ in type B_n corresponding to the subword $(i, i+1, \dots, n-1, n, n-1, \dots, i+1, i)$ of $\mathbf{i}^{\mathsf{BC}}$ is

$$\beta_{i,i} \prec \beta_{i,i+1} \prec \cdots \prec \beta_{i,n} \prec \gamma_{i,n-1} \prec \gamma_{i,n-2} \prec \cdots \prec \gamma_{i,i+1}.$$

Given a Kostant partition $\boldsymbol{c} = (c_\beta)_{\beta \in \Phi^+}$, the string of brackets $S_i^{\mathbf{i}^{\mathsf{BC}}}(\boldsymbol{c})$ is

$$\underbrace{)\cdots)}_{c_{\beta_{1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}}\ \underbrace{)\cdots)}_{c_{\beta_{i,i}}},$$

if $i \neq n$, and is

$$\underbrace{)\cdots)}_{c_{\beta_{1,n}}}\ \underbrace{(\cdots(}_{2c_{\beta_{1,n-1}}}\ \underbrace{)\cdots)}_{2c_{\gamma_{1,n}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,n}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{n-1,n}}}\ \underbrace{(\cdots(}_{2c_{\beta_{n-1,n-1}}}\ \underbrace{)\cdots)}_{2c_{\gamma_{n-1,n}}}\ \underbrace{(\cdots(}_{c_{\beta_{n-1,n}}}\ \underbrace{)\cdots)}_{c_{\beta_{n,n}}},$$

if $i = n$.

Example 5.4. Consider the Kostant partition of type B_3 given by

$$1\ 1\ 1\ 1\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{33\\2\\1}\ \substack{33\\2\\1}\ \substack{2\\33\\2\\1}\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\\2}\ \substack{3\\2}\ \substack{33\\2}\ 3\ 3.$$

We have

$$\begin{array}{rcccccccc} & \substack{2\\1} & 1 & 1 & 1 & 1 & \substack{2\\33\\2\\1} & \substack{33\\2\\1} & \substack{33\\2\\1} \\ S_2: &) & {\color{blue}(} & (& (& \cancel{{\color{red}(}} & \cancel{{\color{red})}} & (& {\color{red}(} \end{array}.$$

Hence, applying f_2 gives

$$1\ 1\ 1\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\end{smallmatrix}\ 3\ 3.$$

Similarly,

$$\begin{array}{l} \qquad \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ 3\ 3 \\ S_3 :\ )\ )\ )\ \cancel{((\ ))}\ ))\ \cancel{(\ (\ (\ )\ )\ )}\ )\ )\ ))\ (\ (\ (\ \cancel{(\ (\ )\ )}\ . \end{array}$$

Hence applying f_3 gives

$$1\ 1\ 1\ 1\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\3\,3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\end{smallmatrix}\ \begin{smallmatrix}3\,3\\2\end{smallmatrix}\ 3\ 3.$$

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \leq i \leq k \leq n$
$\beta_{i,k} = \epsilon_i - \epsilon_{k+1},$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \epsilon_i + \epsilon_k,$	$1 \leq i \leq k \leq n$

TABLE 4. Positive roots of type C_n, expressed both as a linear combination of simple roots and in the canonical realization following [Bou02].

The positive roots for type C_n are listed in Table 4. For $i = 1, \ldots, n-1$ the convex order corresponding to the subword $(i, i+1, \ldots, n-1, n, n-1, \ldots, i+1, i)$ of $\mathbf{i}^{\mathsf{BC}}$ is

$$\beta_{i,i} \prec \beta_{i,i+1} \prec \cdots \beta_{i,n-1} \prec \gamma_{i,i} \prec \gamma_{i,n} \prec \gamma_{i,n-1} \prec \cdots \prec \gamma_{i,i+1}.$$

Given a Kostant partition $\boldsymbol{c} = (c_\beta)_{\beta \in \Phi^+}$ and $i \in I$, the string of brackets $S_i^{\mathbf{i}^{\mathsf{BC}}}(\boldsymbol{c})$ is

$$\underbrace{)\cdots)}_{c_{\beta_{1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}}\ \underbrace{)\cdots)}_{c_{\beta_{i,i}}},$$

if $i \neq n$, and is

$$\underbrace{)\cdots)}_{c_{\gamma_{1,1}}}\ \underbrace{(\cdots(}_{c_{\beta_{1,n-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{1,n}}}\ \underbrace{(\cdots(}_{c_{\gamma_{1,1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\gamma_{n-1,n-1}}}\ \underbrace{(\cdots(}_{c_{\beta_{n-1,n-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{n-1,n}}}\ \underbrace{(\cdots(}_{c_{\gamma_{n-1,n-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{n,n}}},$$

if $i = n$.

Example 5.5. Let $\boldsymbol{c} = (c_\beta)$ be the Kostant partition determined by the **i**-Lusztig datum from Example 3.3; that is, the Kostant partition of type C_3 given by

$$1\ 1\ 1\ 1\ \substack{2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2}\ 3\ 3.$$

To compute f_3, we have

$$\begin{array}{l} \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ 3\ 3 \\ S_3:\ )\ )\ )\ \cancel{(\ )}\ )\ \cancel{(\ (\ (\ )\ )\ )}\ )\ )\ )\ {\color{blue}(}\ (\ (\ \cancel{(\ (\ )\ )}. \end{array}$$

Hence the Kostant partition corresponding to $f_3(\mathbf{c})$ is

$$1\ 1\ 1\ 1\ \substack{2\\1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\,2\\1\,1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2\,2}\ \substack{3\\2}\ {\color{blue}\substack{3\\2}}\ {\color{blue}\substack{3\\2}}\ 3\ 3.$$

Notice that, in accordance with Lemma 4.21, $c_{\alpha_2+\alpha_3}$ has increased by 2.

References

[Bou02] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters* 4–6, Springer-Verlag, Berlin, 2002.

[Bou05] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters* 7–9, Springer-Verlag, Berlin, 2005.

[BZ97] A. Berenstein, A. Zelevinsky, *Total positivity in Schubert varieties*, Comment. Math. Helv. **72** (1997), no. 1, 128–166.

[BZ01] A. Berenstein, A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no. 1, 77–128.

[CT15] J. Claxton, P. Tingley, *Young tableaux, multisegments, and PBW bases*, Sém. Lothar. Combin. **73** (2015), Article B73c.

[Dev16] The Sage Developers, *Sage Mathematics Software* (*Version* 7.4), 2016, `http://www.sagemath.org`.

[HK02] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics, Vol. 42, American Mathematical Society, Providence, RI, 2002.

[Kas91] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[KS97] M. Kashiwara, Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), no. 1, 9–36.

[Kwo] J.-H. Kwon, *A crystal embedding into Lusztig data of type* A, `arXiv:1606.06804` (2016).

[Lec04] B. Leclerc, *Dual canonical bases, quantum shuffles and q-characters*, Math. Z. **246** (2004), no. 4, 691–732.

[Lit98] P. Littelmann, *Cones, crystals, and patterns*, Transform. Groups **3** (1998), no. 2, 145–179.

[Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[Lus10] G. Lusztig, *Introduction to Quantum Groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010.

[LV11] A. D. Lauda, M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math. **228** (2011), no. 2, 803–861.

[Pap94] P. Papi, *A characterization of a special ordering in a root system*, Proc. Amer. Math. Soc. **120** (1994), no. 3, 661–665.

[Rei97] M. Reineke, *On the coloured graph structure of Lusztig's canonical basis*, Math. Ann. **307** (1997), no. 4, 705–723.

[Sai94] Y. Saito, *PBW basis of quantized universal enveloping algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 2, 209–232.

[SCc16] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, 2016, `http://combinat.sagemath.org`.

[Tin17] P. Tingley, *Elementary construction of Lusztig's canonical basis*, in: *Groups, Rings, Group Rings and Hopf Algebras*, dedicated to Don Passman, Contemporary Mathematics **688** (2017), 265–277.

[TW16] P. Tingley, B. Webster, *Mirković–Vilonen polytopes and Khovanov–Lauda–Rouquier algebras*, Compos. Math. **152** (2016), no. 8, 1648–1696.