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Connecting Marginally Large Tableaux and Rigged Configurations via Crystals

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Abstract We show that the bijection from rigged configurations to tensor products of Kirillov-Reshetikhin crystals extends to a crystal isomorphism between the $B(\infty)$ models given by rigged configurations and marginally large tableaux.

Keywords Crystal · Rigged configuration · Marginally large tableaux · Segments

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1 Introduction

In [22, 26], Kerov, Kirillov, and Reshetikhin described a recursive bijection between classically highest-weight rigged configurations in type $A_n^{(1)}$ and standard Young tableaux, showing the Kostka polynomial can be expressed as a fermionic formula. This was then

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extended to Littlewood-Richardson tableaux and classically highest weight elements in a tensor product of Kirillov-Reshetikhin (KR) crystals in [27] for, again, type $A_n^{(1)}$. A similar bijection Φ between rigged configurations and tensor products of the KR crystal $B^{1,1}$ corresponding to the vector representation was extended to all non-exceptional affine types in [35], type $E_6^{(1)}$ in [34], and $D_4^{(3)}$ in [44].

Following [27], it was conjectured that the bijection Φ can be further extended to a tensor product of general KR crystals with the major step being the algorithm for $B^{1,1}$. This has been proven in a variety of cases [37, 38, 42, 44, 45, 47]. Despite this bijection's recursive definition, it is conjectured (see for instance [47]) that Φ sends a combinatorial statistic called cocharge [35] to the algebraic statistic called energy [13], proving the so-called $X = M$ conjecture of [13, 14]. Additionally, the bijection Φ is conjectured to translate the combinatorial R-matrix [21] into the identity on rigged configurations.

The description of Φ on classically highest-weight elements led to a description of classical crystal operators in simply-laced types in [43] and non-simply-laced finite types in [47]. It was shown for type $A_n^{(1)}$ in [8] and $D_n^{(1)}$ in [40] that Φ is a classical crystal isomorphism. Using virtual crystals [36], it can be shown Φ is a classical crystal isomorphism in non-exceptional affine types [47].

Rigged configurations were also extended beyond the context of highest weight classical crystals to $B(\infty)$ in [46]. There is also a similar extension of the Kashiwara-Nakashima tableaux [25] and Kang-Misra tableaux [23] (for type G_2), which are used to describe KR crystals [9, 24, 49], to the marginally large tableaux model of $B(\infty)$ [15, 16]. The goal of this paper is to connect with a crystal isomorphism these two models by using the bijection Φ. In particular, the crystal isomorphism we obtain from Corollary 5.7 is given combinatorially, in the sense that the description does not use the Kashiwara operators.

This paper is organized as follows. In Section 2, we give a background on crystals. In Section 3, we describe the tableaux model for highest weight crystals and marginally large tableaux. In Section 4, we give background on rigged configurations and describe the bijection Φ. In Section 5, we construct our isomorphism between the rigged configuration model and marginally large tableaux model for $B(\infty)$. In Section 6, we describe certain statistics on highest weight crystals and $B(\infty)$.

2 Crystals

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with index set I, Cartan matrix $A = (A_{ab})_{a,b\in I}$, weight lattice P, root lattice Q, fundamental weights $\{\Lambda_a : a \in I\}$, simple roots $\{\alpha_a : a \in I\}$, and simple coroots $\{h_a : a \in I\}$. There is a canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \longrightarrow \mathbf{Z}$ defined by $\langle h_a, \alpha_b\rangle = A_{ab}$, where $P^\vee = \bigoplus_{a\in I} \mathbf{Z} h_a$ is the dual weight lattice. The quantum group associated to $\mathfrak{g}$ is denoted $U_q(\mathfrak{g})$, though we will not need the details concerning $U_q(\mathfrak{g})$. The interested reader is encouraged to see [12, 32] for more details.

An *abstract $U_q(\mathfrak{g})$-crystal* is a nonempty set B together with maps

$$e_a, f_a : B \longrightarrow B \sqcup \{0\}, \quad \varepsilon_a, \varphi_a : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \mathrm{wt} : B \longrightarrow P,$$

subject to the conditions

(1) $f_a b = b'$ if and only if $b = e_a b'$ for $b, b' \in B$ and $a \in I$;
(2) if $f_a b \neq 0$, then $\mathrm{wt}(f_a b) = \mathrm{wt}(b) - \alpha_a$ for all $a \in I$; and
(3) $\varphi_a(b) - \varepsilon_a(b) = \langle h_a, \mathrm{wt}(b)\rangle$ for all $b \in B$ and $a \in I$.

The maps $\{e_a : a \in I\}$ are called the *Kashiwara raising operators* and the maps $\{f_a : a \in I\}$ are called the *Kashiwara lowering operators*.

Example 2.1 For a dominant integral weight λ, the crystal basis

$$B(\lambda) = \{f_{a_k} \cdots f_{a_1} u_\lambda : a_1, \dots, a_k \in I,\ k \in \mathbf{Z}_{\geq 0}\} \setminus \{0\}$$

of an irreducible, highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ is an abstract $U_q(\mathfrak{g})$-crystal. (See [12, 18] for details.) The crystal $B(\lambda)$ is characterized by the following properties.

(1) The element $u_\lambda \in B(\lambda)$ is the unique element such that $\mathrm{wt}(u_\lambda) = \lambda$.
(2) For all $a \in I$, $e_a u_\lambda = 0$.
(3) For all $a \in I$, $f_a^{\langle h_a, \lambda\rangle + 1} u_\lambda = 0$.

Example 2.2 The crystal basis

$$B(\infty) = \{f_{a_k} \cdots f_{a_1} u_\infty : a_1, \dots, a_k \in I,\ k \in \mathbf{Z}_{\geq 0}\}$$

of the negative half $U_q^-(\mathfrak{g})$ of the quantum group (equivalently a Verma module of highest weight 0) is a $U_q(\mathfrak{g})$-crystal. (See [12, 18] for details.) Some important properties of $B(\infty)$ are the following.

(1) The element $u_\infty \in B(\infty)$ is the unique element such that $\mathrm{wt}(u_\infty) = 0$.
(2) For all $a \in I$, $e_a u_\infty = 0$.
(3) For any sequence $(a_1, \dots, a_k)$ from I, $f_{a_k} \cdots f_{a_1} u_\infty \neq 0$.

Let B_1 and B_2 be two abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\psi \colon B_1 \longrightarrow B_2$ is a map $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ such that

(1) $\psi(0) = 0$;
(2) if $b \in B_1$ and $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_a(\psi(b)) = \varepsilon_a(b)$, and $\varphi_a \psi(b)) = \varphi_a(b)$;
(3) for $b \in B_1$, we have $\psi(e_a b) = e_a \psi(b)$ provided $\psi(e_a b) \neq 0$ and $e_a \psi(b) \neq 0$;
(4) for $b \in B_1$, we have $\psi(f_a b) = f_a \psi(b)$ provided $\psi(f_a b) \neq 0$ and $f_i \psi(b) \neq 0$.

A morphism ψ is called *strict* if ψ commutes with e_a and f_a for all $a \in I$. Moreover, a morphism $\psi \colon B_1 \longrightarrow B_2$ is called an *embedding* if the induced map $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ is injective.

Again let B_1 and B_2 be abstract $U_q(\mathfrak{g})$-crystals. The tensor product $B_2 \otimes B_1$ is defined to be the Cartesian product $B_2 \times B_1$ equipped with crystal operations defined by

$$\begin{aligned}
e_a(b_2 \otimes b_1) &= \begin{cases} e_a b_2 \otimes b_1 & \text{if } \varepsilon_a(b_2) > \varphi_a(b_1) \\ b_2 \otimes e_a b_1 & \text{if } \varepsilon_a(b_2) \leq \varphi_a(b_1), \end{cases} \\
f_a(b_2 \otimes b_1) &= \begin{cases} f_a b_2 \otimes b_1 & \text{if } \varepsilon_a(b_2) \geq \varphi_a(b_1) \\ b_2 \otimes f_a b_1 & \text{if } \varepsilon_a(b_2) < \varphi_a(b_1), \end{cases} \\
\varepsilon_a(b_2 \otimes b_1) &= \max\left(\varepsilon_a(b_2), \varepsilon_a(b_1) - \langle h_a, \mathrm{wt}(b_2)\rangle\right) \\
\varphi_a(b_2 \otimes b_1) &= \max\left(\varphi_a(b_1), \varphi_a(b_2) + \langle h_a, \mathrm{wt}(b_1)\rangle\right) \\
\mathrm{wt}(b_2 \otimes b_1) &= \mathrm{wt}(b_2) + \mathrm{wt}(b_1).
\end{aligned}$$

Remark 2.3 Our convention for tensor products is opposite the convention given by Kashiwara in [18].

We say an abstract $U_q(\mathfrak{g})$-crystal is simply a $U_q(\mathfrak{g})$-crystal if it is crystal isomorphic to the crystal basis of an integrable $U_q(\mathfrak{g})$-module.

3 Tableaux Model

Let $\mathfrak{g}$ be of finite classical type or of type G_2. We review the Kashiwara-Nakashima tableaux and Kang-Misra tableaux, which we call *classical tableaux*, model for highest weight crystals $B(\lambda)$ and the marginally large tableaux model for $B(\infty)$.

3.1 Fundamental Crystals and Classical Tableaux

Recall that a tableau is called semistandard over an alphabet $J = \{j_1 \prec j_2 \prec \cdots \prec j_p\}$ if entries are weakly increasing in rows and strictly increasing in columns, with respect to $\prec$. Let $J(X_n)$ be the alphabet for the semistandard tableaux of type X_n. Then

$$\begin{aligned} J(A_n) &= \{1 \prec 2 \prec \cdots \prec n+1\}, \\ J(B_n) &= \{1 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{1}\}, \\ J(C_n) &= \{1 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{1}\}, \\ J(D_{n+1}) &= \{1 \prec \cdots \prec n \prec n+1, \overline{n+1}, \prec \overline{n} \prec \cdots \prec \overline{1}\}, \\ J(G_2) &= \{1 \prec 2 \prec 3 \prec 0 \prec \overline{3} \prec \overline{2} \prec \overline{1}\}. \end{aligned} \tag{3.1}$$

For our purposes, we need only define highest weight crystals for specific fundamental weights. Namely, define a subset $\widehat{P}^+$ of P^+ by

$$\widehat{P}^+ = \begin{cases} \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}\Lambda_n & \text{if } \mathfrak{g} = A_n, C_n, \\ \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}(2\Lambda_n) & \text{if } \mathfrak{g} = B_n, \\ \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}(\Lambda_n + \Lambda_{n+1}) & \text{if } \mathfrak{g} = D_{n+1}, \\ \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\Lambda_2 & \text{if } \mathfrak{g} = G_2. \end{cases}$$

The reason $\widehat{P}^+$ suffices is due to the constructions we will use in what follows. In particular, these weights suffice to define the marginally large tableaux model $\mathcal{T}(\infty)$ of the crystal $B(\infty)$, and thus will be sufficient for us to define our crystal morphism from the rigged configuration model for $B(\infty)$ to $\mathcal{T}(\infty)$. These weights also ensure that we will not have any "spin columns" in types B_n and D_{n+1}.

Recall the fundamental crystals $\mathcal{T}(\Lambda_1)$ in Fig. 1. Next consider some $\lambda \in \widehat{P}^+$; we wish to model an element in $B(\lambda)$. It is from these fundamental crystals that the more general crystals will be defined. For $\lambda \in \widehat{P}^+$ defined by

$$\lambda = \begin{cases} c_1\Lambda_1 + \cdots + c_{n-1}\Lambda_{n-1} + c_n\Lambda_n & \text{if } \mathfrak{g} = A_n, C_n, \\ c_1\Lambda_1 + \cdots + c_{n-1}\Lambda_{n-1} + c_n(2\Lambda_n) & \text{if } \mathfrak{g} = B_n, \\ c_1\Lambda_1 + \cdots + c_{n-1}\Lambda_{n-1} + c_n(\Lambda_n + \Lambda_{n+1}) & \text{if } \mathfrak{g} = D_{n+1}, \\ c_1\Lambda_1 + c_2\Lambda_2 & \text{if } \mathfrak{g} = G_2, \end{cases}$$

let Y_λ be the Young diagram with c_i columns of height i. Define T_λ to be the unique tableau of shape Y_λ such that all entries in the jth row of T_λ are j. We may embed T_λ into $\mathcal{T}(\Lambda_1)^{\otimes|\lambda|}$, where $|\lambda|$ is the number of boxes in Y_λ, by reading the tableaux entries from top-to-bottom starting with the right-most column. Then f_aT_λ, for $a \in I$, is defined using the tensor product rule and the corresponding fundamental crystal above. Now let $\mathcal{T}(\lambda)$ be

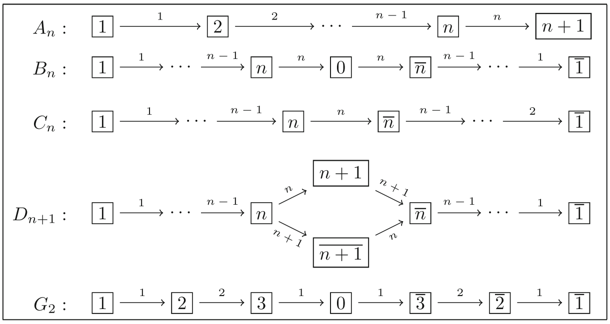


Fig. 1 The fundamental crystals $\mathcal{T}(\Lambda_1)$ when the underlying Lie algebra is of type A_n, B_n, C_n, D_{n+1}, and G_2

the set generated by f_a ($a \in I$) and T_λ. This is the set of *classical tableaux* of shape λ. The description of type A_n, B_n, C_n, D_n tableaux is due to Kashiwara and Nakashima [25] and type G_2 tableaux is due to Kang and Misra [23]. The resulting set is a crystal of semistandard tableaux (with respect to $J(X_n)$) satisfying certain filling conditions. The explicit description of these crystals may be found in [12, 23, 25]. See Fig. 2 for an example.

3.2 Marginally Large Tableaux

Following [7], a semistandard tableau is called *large* if the difference of the number of boxes in the i-th row containing the element i and the total number of boxes in the $(i+1)$-st row is positive. Additionally, following [15], a large (semistandard) tableau is called *marginally large* if the

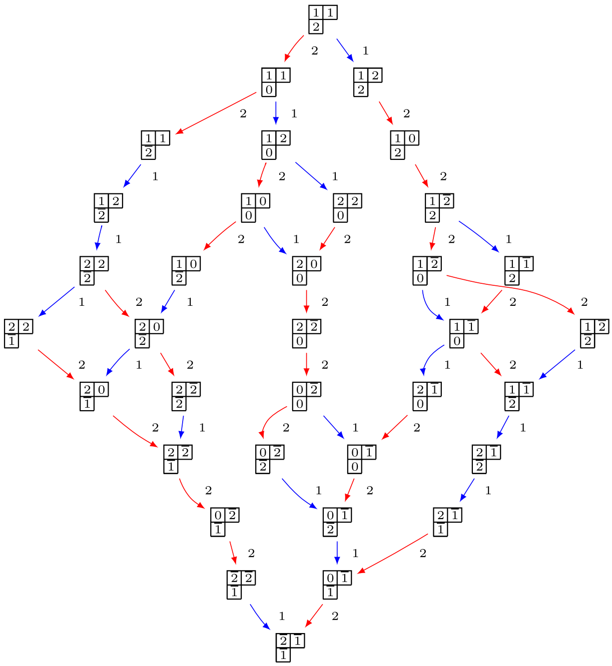


Fig. 2 The crystal graph $\mathcal{T}(\Lambda_1 + 2\Lambda_2)$ of type B_2, created using SageMath [39, 41]

aforementioned difference exactly 1. Such tableaux are defined for simple Lie algebras $\mathfrak{g}$ of type A_n, B_n, C_n, D_{n+1}, and G_2 in [15], and of type E_6, E_7, E_8, and F_4 in [16]. The alphabets over which each tableaux from [15] are defined are given in Eq. 3.1.

The set of marginally large tableaux may be generated through successive application of the Kashiwara lowering operators f_a $(a \in I)$ to a specified highest weight vector. It is in this way that the set of marginally large tableaux work as a combinatorial model for $B(\infty)$. In certain types, additional conditions are required to precisely define the model, so we give the list of conditions for each type-by-type.

Definition 3.1 For $X_n = A_n, B_n, C_n, D_{n+1}, G_2$, define the set $\mathcal{T}(\infty)$ as follows. (By convention, we assume $n = 2$ when $X_n = G_2$.)

(1) The highest weight vector is the unique tableau which consists of $n + 1 - i$ i-colored boxes in the ith row from the top (when written using the English convention).
(2) Each element is marginally large, semistandard with respect to $J(X_n)$, and consists of exactly n rows.

We also have the following additional type-specific requirements.

- $X_n = B_n$ $(n \geq 2)$

  (1) Elements in the ith row are $\preceq \bar{\imath}$.
  (2) A 0-box may occur at most once in a given row.

- $X_n = C_n$ $(n \geq 2)$

  (1) Elements in the ith row are $\preceq \bar{\imath}$.

- $X_n = D_{n+1}$ $(n \geq 3)$

  (1) Elements in the ith row are $\preceq \bar{\imath}$.
  (2) Both $n + 1$ and $\overline{n+1}$ may not appear simultaneously in a single row.

- $X_n = G_2$

  (1) Elements in the second row are $\preceq 3$.
  (2) A 0-box may occur at most once in a given row.

The crystal operators are defined in a similar way as in $\mathcal{T}(\lambda)$. Namely, read entries of a tableau $T \in \mathcal{T}(\infty)$ from top-to-bottom in columns starting with the right-most column to obtain an element of $\mathcal{T}(\Lambda_1)^{\otimes N}$, where N is the number of boxes in T. Then apply the tensor product rule to obtain $f_a T$ and $e_a T$, $a \in I$. The difference in $\mathcal{T}(\infty)$ is that the action of f_a may involve the addition of columns and the action of e_a may involve the removal of columns. See [15] for details.

Theorem 3.2 ([15]) *Let $\mathfrak{g}$ be a simple Lie algebra of finite type X_n. Then $\mathcal{T}(\infty) \cong B(\infty)$ as $U_q(\mathfrak{g})$-crystals.*

Example 3.3 Consider type A_3. The top part of the crystal graph $\mathcal{T}(\infty)$ is shown in Fig. 3 down to depth 3. The notation used at the vertices is condensed so that all place holding i-boxes in the ith row are removed. If the resulting reduction yields a row with no boxes, then that row appears with one box containing $*$. The graph in the figure is modified output from SageMath [39, 41].

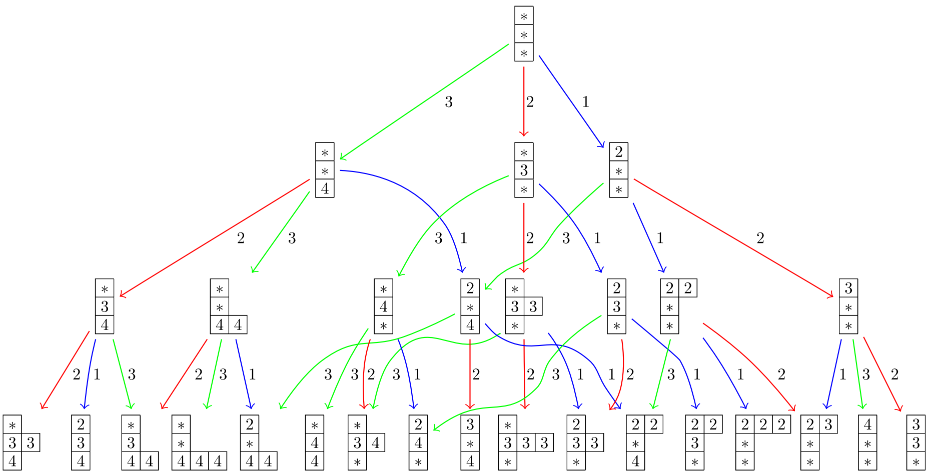


Fig. 3 The top part of $\mathcal{T}(\infty)$ for type A_3

Following [16], we call a column of any tableau T a *basic column* if it has height r and is filled with $(1, \ldots, r)$. From [15], consider the set

$$\mathcal{T}^L := \left\{ T \in \bigcup_{\lambda \in \widehat{P}_+} \mathcal{T}(\lambda) : T \text{ is large} \right\}$$

We may partition $\mathcal{T}^L$ into equivalence classes by saying $T \sim T'$ if they differ only by basic columns. Note that $f_a T \neq 0$ for all $T \in \mathcal{T}^L$. If $f_a T$ is large, then for all $S \in [T]$ such that $f_a S$ is large, we have $f_a S \in [f_a T]$. In other words, the crystal operators essentially preserve equivalence classes. Moreover, if $f_a S \notin [f_a T]$, then $f_a S$ differs from a unique element in $[f_a T]$ only by adding a single basic column of height a. Additionally, every equivalence class has exactly one marginally large tableaux. The details of these statements can be found in [15].

4 Rigged Configurations

4.1 Crystal Structure

We first need to consider an affine type $\widetilde{\mathfrak{g}}$ whose classical subalgebra is $\mathfrak{g}$. However we do not do so in the usual fashion by taking the untwisted affine algebra, but instead consider those given by Table 1.

Set $\mathcal{H} = I \times \mathbf{Z}_{>0}$. Consider a multiplicity array

$$L = \left(L^{(a)} \in \mathbf{Z}_{\geq 0} : a \in I \right)$$

Table 1 The association of affine type $\widetilde{\mathfrak{g}}$ with a classical type $\mathfrak{g}$ used here

$\mathfrak{g}$	A_n	B_n	C_n	D_{n+1}	G_2
$\widetilde{\mathfrak{g}}$	$A_n^{(1)}$	$D_{n+1}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(1)}$	$D_4^{(3)}$

and a dominant integral weight λ for $\mathfrak{g}$. We call a sequence of partitions $\nu = \{\nu^{(a)} : a \in I\}$ an (L, λ)-*configuration* if

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{a\in I} L^{(a)} \Lambda_a - \lambda, \tag{4.1}$$

where $m_i^{(a)}$ is the number of parts of length i in the partition $\nu^{(a)}$ and $\{\alpha_a : a \in I\}$ are the simple roots for $\mathfrak{g}$. The set of all such (L, λ)-configurations is denoted $C(L, \lambda)$. To an element $\nu \in C(L, \lambda)$, define the *vacancy numbers* of ν to be

$$p_i^{(a)}(\nu) = p_i^{(a)} = L^{(a)} - \sum_{(b,j)\in\mathcal{H}_0} A_{ab} \min(i, j) m_j^{(b)}. \tag{4.2}$$

Recall that a partition is a multiset of integers (typically sorted in weakly decreasing order). More generally, a *rigged partition* is a multiset of pairs of integers (i, x) such that $i > 0$ (typically sorted under weakly decreasing lexicographic order). Each (i, x) is called a *string*, while i is called the length or size of the string and x is the *rigging*, *label*, or *quantum number* of the string. Finally, a *rigged configuration* is a pair (ν, J) where $\nu \in C(L, \lambda)$ and $J = (J_i^{(a)} : (a, i) \in \mathcal{H})$, where each $J_i^{(a)}$ is a weakly decreasing sequence of riggings of strings of length i in $\nu^{(a)}$. We call a rigged configuration *valid* if every label $x \in J_i^{(a)}$ satisfies the inequality $p_i^{(a)} \geq x$ for all $(a, i) \in \mathcal{H}$. We say a rigged configuration is *highest weight* if $x \geq 0$ for all labels x. Define the *colabel* or *coquantum number* of a string (i, x) to be $p_i^{(a)} - x$. For brevity, we will often denote the ath part of (ν, J) by $(\nu, J)^{(a)}$ (as opposed to $(\nu^{(a)}, J^{(a)})$).

Definition 4.1 Let $(\nu_\emptyset, J_\emptyset)$ be the rigged configuration with empty partition and empty riggings and let L be the multiplicity array of all zeros. Define $\mathrm{RC}(\infty)$ to be the graph generated by $(\nu_\emptyset, J_\emptyset)$, e_a, and f_a, for $a \in I$, where e_a and f_a acts on elements (ν, J) in $\mathrm{RC}(\infty)$ as follows. Fix $a \in I$ and let x be the smallest label of $(\nu, J)^{(a)}$.

e_a: If $x \geq 0$, then set $e_a(\nu, J) = 0$. Otherwise, let ℓ be the minimal length of all strings in $(\nu, J)^{(a)}$ which have label x. The rigged configuration $e_a(\nu, J)$ is obtained by replacing the string (ℓ, x) with the string $(\ell - 1, x + 1)$ and changing all other labels so that all colabels remain fixed.

f_a: If $x > 0$, then add the string $(1, -1)$ to $(\nu, J)^{(a)}$. Otherwise, let ℓ be the maximal length of all strings in $(\nu, J)^{(a)}$ which have label x. Replace the string (ℓ, x) by the string $(\ell + 1, x - 1)$ and change all other labels so that all colabels remain fixed.

The remaining crystal structure on $\mathrm{RC}(\infty)$ is given by

$$\varepsilon_a(\nu, J) = \max\{k \in \mathbf{Z}_{\geq 0} : e_a^k(\nu, J) \neq 0\}, \tag{4.3a}$$

$$\varphi_a(\nu, J) = \varepsilon_a(\nu, J) + \langle h_a, \mathrm{wt}(\nu, J)\rangle, \tag{4.3b}$$

$$\mathrm{wt}(\nu, J) = -\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = -\sum_{a\in I} |\nu^{(a)}| \alpha_a. \tag{4.3c}$$

It is worth noting that, in this case, the definition of the vacancy numbers reduces to

$$p_i^{(a)}(\nu) = p_i^{(a)} = -\sum_{(b,j)\in\mathcal{H}_0} A_{ab} \min(i, j) m_j^{(b)}. \tag{4.4}$$

Example 4.2 Rigged configurations will be shown as a sequence of partitions where the vacancy numbers will be written on the left and the corresponding rigging on the right. Let $\mathfrak{g}$ be of type A_5 and $(\nu, J) = f_5 f_4 f_5 f_2 f_1 f_2 f_3 (\nu_\emptyset, J_\emptyset)$ be the rigged configuration

$$(\nu, J) = -1 \,\square\, -1 \quad -2 \,\square\square\, -1 \quad 0 \,\square\, 1 \quad 0 \,\square\, -1 \quad -3 \,\square\square\, -1$$

Then $\mathrm{wt}(\nu, J) = -\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - 2\alpha_5$,

$$e_2(\nu, J) = -1 \,\square\, -1 \quad 0 \,\square\, 0 \quad 0 \,\square\, 1 \quad 0 \,\square\, -1 \quad -3 \,\square\square\, -1$$

and

$$f_2(\nu, J) = -1 \,\square\, -1 \quad -4 \,\square\square\square\, -2 \quad 0 \,\square\, 1 \quad 0 \,\square\, -1 \quad -3 \,\square\square\, -1$$

Theorem 4.3 ([46]) *The map defined by $(\nu_\emptyset, J_\emptyset) \mapsto u_\infty$, where u_∞ is the highest weight element of $B(\infty)$, is a $U_q(\mathfrak{g})$-crystal isomorphism* $\mathrm{RC}(\infty) \cong B(\infty)$.

We can extend the crystal structure on rigged configurations to model $B(\lambda)$ as follows. Consider a multiplicity array L such that $\lambda = \sum_{a \in I} L^{(a)} \Lambda_a$. We first we modify the definition of the weight to be $\mathrm{wt}'(\nu, J) = \mathrm{wt}(\nu, J) + \lambda$. Next, modify the crystal operators by saying $f_a(\nu, J) = 0$ if $\varphi_a(\nu, J) = 0$. Equivalently, we say $f_a(\nu, J) = 0$ if the result under f_a above is not a valid rigged configuration. Let $\mathrm{RC}(\lambda)$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under these modified crystal operators. This arises from the natural projection of $B(\infty) \longrightarrow B(\lambda)$. See Fig. 4 for an example of $\mathrm{RC}(\lambda)$.

Theorem 4.4 ([47]) *Let $\mathfrak{g}$ be of finite type. Then* $\mathrm{RC}(\lambda) \cong B(\lambda)$.

4.2 Bijection with Tableau Model for Highest Weight Crystals

Kirillov-Reshetikhin (KR) crystals are crystals $B^{r,s}$ associated to certain finite-dimensional $U_q([\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}])$-modules $W^{r,s}$, where r is a node in the Dynkin diagram and s is a positive integer. As $U_q(\mathfrak{g})$-crystals, KR crystals have the direct sum decompositions

$$B^{r,s} = B(s\Lambda_r) \oplus \bigoplus_{\lambda} B(\lambda),$$

where the sum is over $\lambda \in P^+$ satisfying certain conditions [6, 9–11, 13, 14, 33]. We note that we only work with the classical crystal structure of $B^{r,s}$, and as such, we simply consider $B^{r,s}$ to be a $U_q(\mathfrak{g})$-crystal.

In [35], a bijection Φ from classically highest weight elements in a tensor product of KR crystals of the form $(B^{1,1})^{\otimes N}$ for all non-exceptional affine types was described. A similar bijection in type $E_6^{(1)}$ and $D_4^{(3)}$ was given in [34] and [44], respectively. For simplicity, if $B = \bigotimes_{i=1}^{N} B^{r_i,1}$ we write $\mathrm{RC}(B) = \mathrm{RC}(\lambda)$, where $\lambda = \sum_{a \in I} c_a \Lambda_a$ with c_a equal to the number of factors $B^{a,1}$ occurring in B.

Remark 4.5 We define

$$\widetilde{B}^{r,1} = \begin{cases} B^{n,2} & \mathfrak{g} = B_n \text{ and } r = n, \\ B^{n,1} \otimes B^{n+1,1} & \mathfrak{g} = D_{n+1} \text{ and } r = n, \\ B^{r,1} & \text{otherwise.} \end{cases}$$

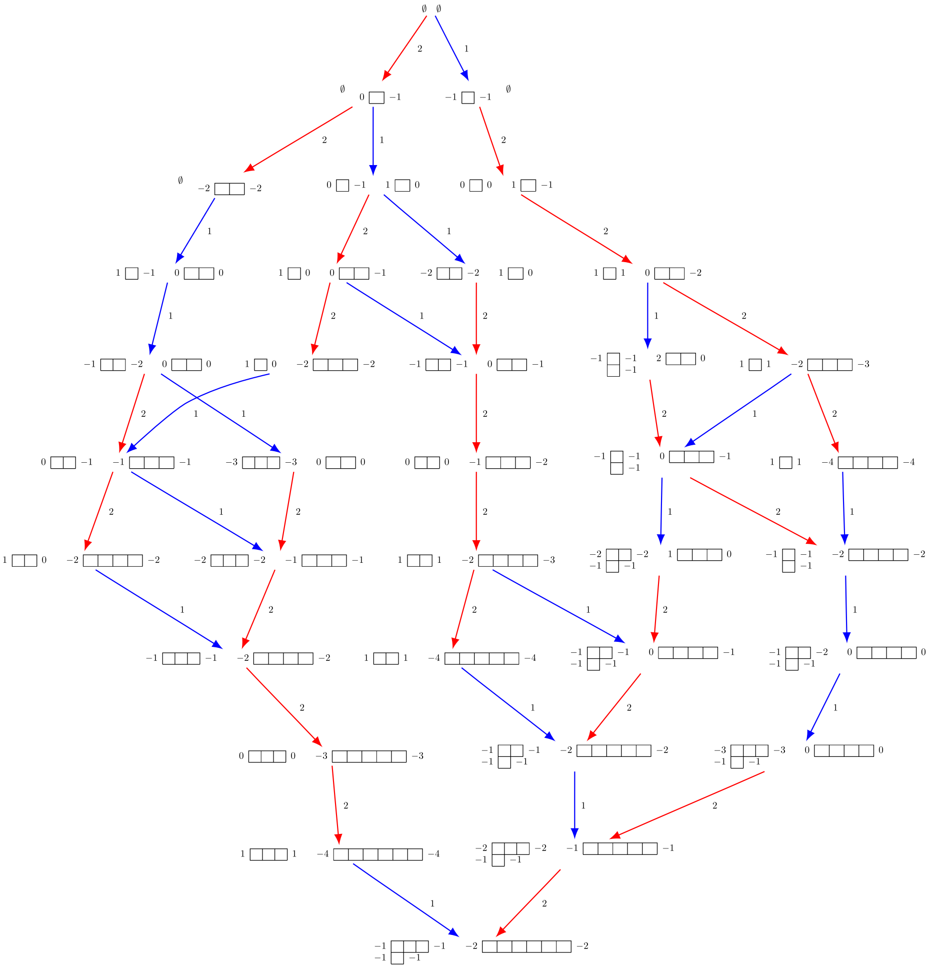


Fig. 4 The crystal graph $\mathrm{RC}(\Lambda_1 + 2\Lambda_2)$ of type B_2, created using SageMath [39, 41]

Because we will only use $\widetilde{B}^{r,s}$ for the remainder of this paper and to ease the burden of notation, we will simply write $B^{r,s}$. We also note that this allows us to not consider any special modifications to Φ as in [47].

The bijection Φ is given by applying the basic algorithm given in [35]

$$\delta\colon \mathrm{RC}(B^{1,1} \otimes B^*) \longrightarrow \mathrm{RC}(B^*)$$

as many times as necessary, where $B^* = \bigotimes_{i=1}^{N} B^{r_i,1}$. The algorithm δ is given by traversing the crystal graph $\mathcal{T}(\Lambda_1)$ (of type $\mathfrak{g}$) starting at $\boxed{1} \in \mathcal{T}(\Lambda_1)$, where for each edge a we remove a box from a singular string from $\nu^{(a)}$ of strictly longer length than the previously selected string after removal. For $\widetilde{\mathfrak{g}}$ of type $D_{n+1}^{(1)}$, we choose the smaller singular string between $\nu^{(n)}$ and $\nu^{(n+1)}$ when we are at $\boxed{n-1} \in \mathcal{T}(\Lambda_1)$. If we cannot find a singular string or there are no outgoing arrows when we are at $\boxed{r} \in \mathcal{T}(\Lambda_1)$, then we say δ returns r and we make all changed strings singular.

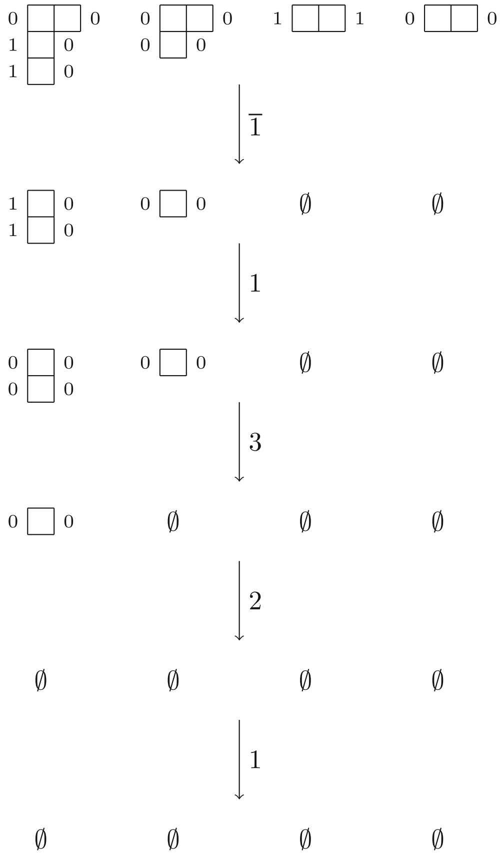
We also have the following modification for $\widetilde{\mathfrak{g}}$ of type $D_{n+1}^{(2)}$. Let $\ell^{(a)}$ denote the original length of the selected strings in $\nu^{(a)}$ (for $a < r$, we have $\ell^{(a)} = 1$). We say a string (i, x) is *quasi-singular* if $x = p_i^{(n)} - 1$ and there does not exist a singular string of length i. For $\nu^{(n)}$, if the singular string has length 1, we immediately return $\emptyset \in B^{1,1}$. Otherwise we look for the smallest string of longer length than the previously which is either

(S) singular,
(Q) quasi-singular.

If no such string exists, we return $n-1$ (as usual). If we are in case (S), we remove 2 boxes from the singular string and proceed from $\boxed{\overline{n-1}} \in \mathcal{T}(\Lambda_1)$. If we are in case (Q), we remove a box from the quasi-singular string and look for a larger singular string in $\nu^{(n)}$. If no such string exists, we return 0. Otherwise we say we are in case (Q,S) and remove a box from the found singular string. We then continue from $\boxed{\overline{n-1}} \in \mathcal{T}(\Lambda_1)$. If we are at $\boxed{\overline{\imath}} \in \mathcal{T}(\Lambda_1)$ and the length of the previously selected string before removal equals $\ell^{(a)}$, we remove a second box from the string originally selected in $\nu^{(a)}$.

After all boxes are removed, we make all of the changed strings singular unless we are in case (Q,S), in which case the (longer) selected singular string in $\nu^{(n)}$ is made quasi-singular.

Example 4.6 Consider $B = (B^{1,1})^{\otimes 5}$ of type $D_5^{(2)}$. Therefore applying δ each time, we have



(where the result from each application of δ is to the right of the arrow) and resulting in

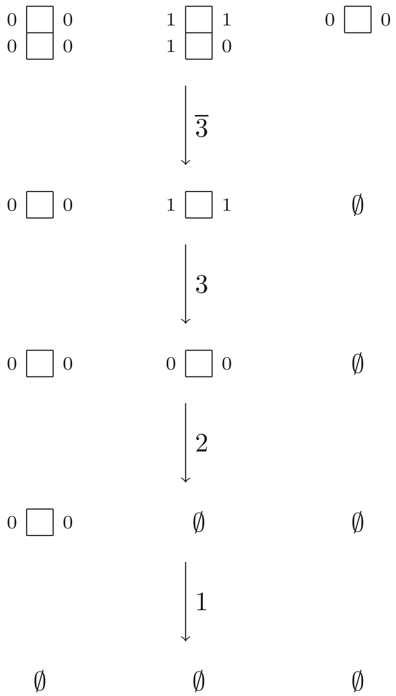
$$\boxed{\overline{1}} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1}\,.$$

We can now extend this bijection to $\bigotimes_{i=1}^{N} B^{r_i,1}$ as given for types $A_n^{(1)}$ [27], $D_{n+1}^{(1)}$ [42], $A_{2n-1}^{(2)}$ [47], $D_{n+1}^{(2)}$ [36], and $D_4^{(3)}$ [44] by considering the map

$$\mathrm{lt}\colon\ \mathrm{RC}(B^{r,1} \otimes B^{*}) \longrightarrow \mathrm{RC}(B^{1,1} \otimes B^{r-1,1} \otimes B^{*})$$

for $r \geq 2$ which adds a singular string of length 1 to all $\nu^{(a)}$ for $a < r$ and then applying δ. We can combine these two steps $\delta' := \delta \circ \mathrm{lt}$ where we just begin δ starting at $\boxed{r} \in \mathcal{T}(\Lambda_1)$ (i.e., the first box we try to remove is in $\nu^{(r)}$). Unless otherwise noted, we will be using δ' in place of δ.

Example 4.7 Consider $B^{1,1} \otimes B^{2,1} \otimes B^{1,1}$ of type $A_5^{(2)}$. Therefore applying δ each time, we have



and resulting in

$$\boxed{\overline{3}} \otimes \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} \otimes \boxed{1}\,.$$

For $\delta^{-1}(b)$, in general we select the largest singular strings starting at $\boxed{b} \in \mathcal{T}(\Lambda_1)$ and following arrows in reverse until we reach $\boxed{r} \in \mathcal{T}(\Lambda_1)$.

Remark 4.8 We note that for $B(\Lambda_r) \subseteq B^{r,1}$, the classical tableaux representation is the same as the Kirillov–Reshetikhin tableaux representation of [38, 42, 44, 47].

We also have the following from [47, Prop. 6.4] by using the results of [8, 36, 40, 47].

Theorem 4.9 *Let* $B = \bigotimes_{i=1}^{N} B^{r_i,1}$ *be of type* $\widetilde{\mathfrak{g}}$. *Then* Φ *is a* $U_q(\mathfrak{g})$*-crystal isomorphism, where* $\widetilde{\mathfrak{g}}$ *and* $\mathfrak{g}$ *are related via Table 1.*

Given dominant integral weight $\lambda = \sum_{a \in I} c_a \Lambda_a \in \widehat{P}^+$, define

$$B^{\otimes\lambda} = \bigotimes_{a \in I} (B^{a,1})^{\otimes c_a},$$

where the factors are ordered to be weakly decreasing with respect to a.

Example 4.10 If $\lambda = 2\Lambda_4 + 3\Lambda_1$ in type A_{17}, then $B^{\otimes\lambda} = (B^{4,1})^{\otimes 2} \otimes (B^{1,1})^{\otimes 3}$.

Next we can restrict Φ to a (classical, or $U_q(\mathfrak{g})$-)crystal isomorphism between $\mathrm{RC}(\lambda)$ and $\mathcal{T}(\lambda)$ as follows. We recall that there exists, by weight considerations, a unique copy

$$B\left(\sum_{i=1}^{N} \mu_i\right) \subseteq \bigotimes_{i=1}^{N} B(\mu_i)$$

generated by $u_{\mu_1} \otimes \cdots \otimes u_{\mu_N}$, where $u_{\mu_i} \in B(\mu_i)$ is the unique highest weight element. Then there exists a unique embedding

$$\mathcal{T}(\lambda) \subseteq \bigotimes_{a \in I} \mathcal{T}(\Lambda_a)^{\otimes c_a} \subseteq B^{\otimes\lambda}. \tag{4.5}$$

Because we have chosen the ordering of $B^{\otimes\lambda}$ to be in decreasing order, the natural embedding of $\mathcal{T}(\lambda) \longrightarrow \mathcal{T}(\Lambda_1)^{\otimes|\lambda|}$ agrees with the natural classical embedding of $B^{\otimes\lambda} \longrightarrow \mathcal{T}(\Lambda_1)^{\otimes|\lambda|}$. Moreover the highest weight element $T_\lambda \in \mathcal{T}(\lambda)$ is given by (a tensor product of) basic columns. So in $\mathrm{RC}(B^{\otimes\lambda})$, considered as a $U_q(\mathfrak{g})$-crystal, the unique connected component generated by $(\nu_\emptyset, J_\emptyset)$ is $\mathrm{RC}(\lambda)$. Therefore $\Phi(\nu_\emptyset, J_\emptyset) = T_\lambda$, and hence we have the following.

Proposition 4.11 *The crystal isomorphism* $\Phi\colon \mathrm{RC}(B^{\otimes\lambda}) \longrightarrow B^{\otimes\lambda}$ *restricts to a crystal isomorphism between* $\mathrm{RC}(\lambda)$ *and* $\mathcal{T}(\lambda)$.

5 The Crystal Isomorphism Between RC(∞) and $\mathcal{T}(\infty)$

Let $\mathfrak{g}$ be of type A_n, B_n, C_n, D_{n+1}, or G_2. By knowing to which $B(\lambda)$ a particular rigged configuration belongs (in fact, there are infinite such λ), we can extend the bijection between rigged configurations and tensor products of KR crystals to $B(\infty)$ by projecting down to $B(\lambda)$. We show that this implies the (induced) map given by lifting the isomorphism $\Phi\colon \mathrm{RC}(\lambda) \longrightarrow \mathcal{T}(\lambda)$ from Proposition 4.11 is an isomorphism between $\mathrm{RC}(\infty)$ and $\mathcal{T}(\infty)$ (and thus the unique isomorphism between $\mathrm{RC}(\infty)$ and $\mathcal{T}(\infty)$ since the automorphism group of $B(\infty)$ is the trivial group).

Consider

$$\mathrm{RC}^V = \bigsqcup_{\lambda \in P^+} \mathrm{RC}(\lambda),$$

and denote $(\nu, J, \lambda) \in \mathrm{RC}^V$ as the element $(\nu, J) \in \mathrm{RC}(\lambda)$. Define an equivalence relation on RC^V by asserting

$$(\nu, J, \lambda) \sim (\nu', J', \lambda') \iff \nu = \nu' \text{ and } J = J'. \tag{5.1}$$

Note the vacancy numbers will vary over the equivalence class. Equivalently, we have defined a subset $W_{(\nu,J)} \subseteq P^+$ such that $(\nu, J) \in \mathrm{RC}(\lambda)$ for all $\lambda \in W_{(\nu,J)}$. We show that every equivalence class of large tableaux embeds into an equivalence class of RC^V and that Φ induces a bijection from $\mathrm{RC}^V / \sim$ to $\mathcal{T}(\infty) = \mathcal{T}^L / \sim$. Subsequently, we show that this induced bijection is the desired crystal isomorphism.

For a sequence of partitions $\nu = (\nu^{(a)})_{a \in I}$, define $\lambda_\nu \in P^+$ by

$$\lambda_\nu := \sum_{\substack{a \in I \\ a < n}} \left(|\nu^{(a)}| + 1 \right) \Lambda_a + \lambda_\nu^{(n)},$$

where

$$\lambda_\nu^{(n)} := \begin{cases} 2\left(|\nu^{(n)}| + 1\right) \Lambda_n & \mathfrak{g} = B_n, \\ \left(\max\left(|\nu^{(n)}|, |\nu^{(n+1)}|\right) + 1\right)(\Lambda_n + \Lambda_{n+1}) & \mathfrak{g} = D_{n+1}, \\ \left(|\nu^{(n)}| + 1\right) \Lambda_n & \text{otherwise,} \end{cases}$$

and

$$\mathrm{RC}^{EV} = \{(\nu, J, \lambda) \in \mathrm{RC}^V : \lambda \geq \lambda_\nu\}.$$

Here, $\lambda \leq \mu$ means $\lambda = (\lambda_i : i \in I)$ and $\mu = (\mu_i : i \in I)$ with $\lambda_i \leq \mu_i$ for all $i \in I$. We can restrict the equivalence relation given in Eq. 5.1 to RC^{EV} (so that classes in RC^{EV} are subclasses of those in RC^V). Call a rigged configuration *extra valid* if it belongs to RC^{EV}, and call a rigged configuration *marginally extra valid* if $(\nu, J) \in \mathrm{RC}(\lambda_\nu)$. We note that for each equivalence class in RC^{EV}, there is a unique marginally extra valid rigged configuration because it has the smallest possible vacancy numbers.

Lemma 5.1 *If* $(\nu, J) \in \mathrm{RC}(\infty)$*, then* $(\nu, J) \in \mathrm{RC}(\lambda_\nu)$.

Proof This clearly holds for $(\nu_\emptyset, J_\emptyset) \in \mathrm{RC}(0)$. We will now proceed by induction by applying f_a for some $a \in I$. Suppose $(\nu, J) \in \mathrm{RC}(\lambda_\nu)$, we will show that $(\nu', J') = f_a(\nu, J)$ is in $\mathrm{RC}(\lambda_{\nu'})$. We note that the only possible failure will occur if $x' > p_{i+1}^{(a)}(\nu')$ for the string (i, x) acted on by f_a since all other colabels remain fixed. We have

$$p_{i+1}^{(a)}(\nu') - p_{i+1}^{(a)}(\nu) = -1$$

since $\lambda_{\nu'} - \lambda_\nu = \Lambda_a$. But because $x' - x = -1$, we must have $x' \leq p_{i+1}^{(a)}(\nu')$. Therefore $(\nu', J') \in \mathrm{RC}(\lambda_{\nu'})$. Since there is some path to $(\nu_\emptyset, J_\emptyset)$, the proof follows by induction. □

Lemma 5.2 *Let* $T \in \mathcal{T}^L$. *Then* $\Phi^{-1}(T) \in \mathrm{RC}^V$. *Moreover, if* $t \in [T]$, *then* $\Phi^{-1}(t) \in [\Phi^{-1}(T)]$.

Proof Fix a large tableau T. By the definition of Φ and RC^V, we have $\Phi^{-1}(T) \in \mathrm{RC}^V$. Next we note a column in T of height r has the form

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline r-1 \\ \hline x \\ \hline \end{array} \tag{5.2}$$

where $1 \leq r \leq n$. We are going to add a column of the form above in $B^{r,1}$. Suppose $B = \bigotimes_{i=1}^{N} B^{r_i,s_i}$ and $(\nu, J) \in \mathrm{RC}(B^{a-1,1} \otimes B)$ for $a < r$. Applying δ^{-1} to the column in Eq. 5.2 will change (ν, J), and the order by which (ν, J) is affected is determined by reading the column from top to bottom. Indeed, applying δ^{-1} corresponding to the a-box of the column will add 1 to the vacancy numbers of $\nu^{(a)}$ and subtract 1 from the vacancy numbers of $\nu^{(a-1)}$ if $a > 1$. Now suppose we are performing δ^{-1} corresponding to the x-box at the bottom of the column. Then this application of δ^{-1} can only add boxes to $\nu^{(a)}$ for $a \geq r$, and can at most decrease the vacancy numbers in $\nu^{(r-1)}$ by 1. In particular, if $x = r$, then the net result of adding this column is that the vacancy numbers of $\nu^{(r)}$ increased by 1 and the vacancy numbers of $\nu^{(a)}$ for $a < r$ are left unchanged.

In applying Φ^{-1} to T, we are moving from right to left in T, so we are applying δ^{-1} to columns weakly increasing in height. Moreover, t differs from T by the, without loss of generality, addition of columns with $x = r$. Therefore from the above, we must have $\Phi^{-1}(t) \sim \Phi^{-1}(T)$ for any $t \in [T]$. □

Lemma 5.3 *Let* $(\nu, J) \in \mathrm{RC}^{EV}$. *Then* $\Phi(\nu, J) \in \mathcal{T}^L$. *Moreover if* $(\nu', J') \in [(\nu, J)]$, *then* $\Phi(\nu', J') \in [\Phi(\nu, J)]$.

Proof Fix some extra valid rigged configuration (ν, J). Suppose $\Phi(\nu, J) \notin \mathcal{T}^L$. Therefore during the procedure of Φ in a column of height r at height $a < r$, we return $x \geq a$, so we remove at least one box from $\nu^{(a)}$. Therefore we must remove at least

$$1 + \langle \alpha_a, \lambda_\nu \rangle = 1 + 1 + |\nu^{(a)}|$$

boxes from $\nu^{(a)}$ since we must return at least x by the semistandard condition. This is a contradiction, and so we must return a. Similarly for the left-most column of height r, we must return r. Hence $\Phi(\nu, J) \in \mathcal{T}^L$.

Next we show $\Phi(\nu', J') \in [\Phi(\nu, J)]$. Consider $(\nu', J') \in \mathrm{RC}(\lambda_{\nu'})$ such that $\lambda_{\nu'} - \lambda_\nu = \Lambda_r$ for some $1 \leq r \leq n$. We will show that $\Phi(\nu', J')$ differs from $\Phi(\nu, J)$ by a basic column of height r. We note that

$$p_i^{(a)}(\nu') - p_i^{(a)}(\nu) = \delta_{ar}, \tag{5.3}$$

and therefore all columns of height at least $r+1$ are equal under Φ. That is δ returns the same elements on both (ν', J') and (ν, J). Furthermore, once we've removed all such columns (there are the same number of columns in each), the results are equivalent such that the difference of the weights is still Λ_r. Hence Eq. 5.3 still holds.

Now we have one additional column of height r in $\lambda_{\nu'}$. From Eq. 5.3, we must have all strings of $\nu'^{(r)}$ being non-singular. Therefore δ returns r, and we increase all vacancy numbers of $\nu'^{(r-1)}$. Thus all strings of $\nu'^{(r-1)}$ are non-singular and iterating this, we see that we return a basic column of height r.

At this point, the resulting rigged configurations are equal (not just equivalent as they have the same weight), and hence the remaining result from Φ are equal. Since there exists a unique element of minimal weight in $[(\nu, J)]$, the claim follows by induction. □

The following lemma is analogous to [15, Lemma 3.2], which shows that the crystal operators are well-defined on equivalence classes.

Lemma 5.4 *Fix $a \in I$.*

(1) *If $(\nu, J) \in \mathrm{RC}^{EV}$, then $f_a(\nu, J) \neq 0$.*
(2) *Given any element of $\mathrm{RC}(\infty)$, we can always find a representative $(\nu, J) \in \mathrm{RC}^{EV}$ such that $f_a(\nu, J)$ is a valid rigged configuration.*
(3) *If (ν, J) and (ν', J') are in the same equivalence class in $\mathrm{RC}^V / \sim$, then $[f_a(\nu, J)] = [f_a(\nu', J')]$.*
(4) *If (ν, J) is valid, then $e_a(\nu, J)$ is either valid or zero.*
(5) *If (ν, J) and (ν', J') are in the same equivalence class in $\mathrm{RC}^V / \sim$, then either $[e_a(\nu, J)] = [e_a(\nu', J')]$ or both $e_a(\nu, J) = e_a(\nu', J') = 0$.*

Proof These statements can be seen directly from the definitions. □

Thus we can define

$$e_a[(\nu, J)] = [e_a(\nu, J)] \tag{5.4a}$$
$$f_a[(\nu, J)] = [f_a(\nu, J)] \tag{5.4b}$$
$$\mathrm{wt}[(\nu, J)] = \sum_{a \in I} |\nu^{(a)}| \Lambda_a, \tag{5.4c}$$
$$\varepsilon_a[(\nu, J)] = \max\{e_a^k(\nu, J) \neq 0 : k \in \mathbf{Z}_{>0}\}, \tag{5.4d}$$
$$\varphi_a[(\nu, J)] = \varepsilon_a[(\nu, J)] + \langle \mathrm{wt}[(\nu, J)], h_a \rangle, \tag{5.4e}$$

for any $[(\nu, J)] \in \mathrm{RC}^V / \sim$ with appropriate representative (ν, J). Therefore a straightforward check shows the following.

Proposition 5.5 *Equation 5.4 defines an abstract $U_q(\mathfrak{g})$-crystal structure on $\mathrm{RC}^V / \sim$.*

Define a map $\Psi\colon \mathrm{RC}(\infty) \longrightarrow \mathcal{T}(\infty)$ using the sequence of maps

$$\begin{array}{ccccccccc} \mathrm{RC}(\infty) & \twoheadrightarrow & \mathrm{RC}(\lambda_\nu) & \hookrightarrow & B^{\otimes \lambda_\nu} & \twoheadrightarrow & \mathcal{T}(\lambda_\nu) & \hookrightarrow & \mathcal{T}(\infty), \\ (\nu, J) & \mapsto & (\nu, J) & \mapsto & (\nu, J) & \mapsto & \Phi(\nu, J) & \mapsto & \Phi(\nu, J). \end{array}$$

Conversely, for $T \in \mathcal{T}(\infty)$, let $\lambda_T \in P^+$ partition shape of T. There is a natural crystal embedding of a tableau T into a tensor product of its columns in $B^{\otimes \lambda_T}$. Denote the image of T in $B^{\otimes \lambda_T}$ by $T^{\otimes \lambda_T}$. Now define a map $\Xi \colon \mathcal{T}(\infty) \longrightarrow \mathrm{RC}(\infty)$ by the sequence of maps

$$\begin{array}{ccccccccc} \mathcal{T}(\infty) & \twoheadrightarrow & \mathcal{T}(\lambda_T) & \hookrightarrow & B^{\otimes \lambda_T} & \twoheadrightarrow & \mathrm{RC}(\lambda_T) & \hookrightarrow & \mathrm{RC}(\infty), \\ T & \mapsto & T & \mapsto & T^{\otimes \lambda_T} & \mapsto & \Phi^{-1}(T^{\otimes \lambda_T}) & \mapsto & \Phi^{-1}(T^{\otimes \lambda_T}). \end{array}$$

Theorem 5.6 *We have* $\Xi \circ \Psi = \mathrm{id}_{\mathrm{RC}(\infty)}$ *and* $\Psi \circ \Xi = \mathrm{id}_{\mathcal{T}(\infty)}$.

Proof Given a marginally large tableaux T of shape λ_T, begin by projecting down to $B(\lambda_T)$. This preserves the tableaux T and consider the natural embedding T' in $\bigotimes_i B^{r_i,1}$ given by Eq. 4.5. Next take $\Phi(T')$, recall from Theorem 4.9 that Φ is a bijection, and lift the resulting rigged configuration to $\mathrm{RC}(\infty)$. Last, we note that this procedure is well-defined over the equivalence class of large tableaux by Lemma 5.2 and Lemma 5.3, so this gives us the desired bijection. □

Corollary 5.7 *The bijection* Ψ *is a crystal isomorphism.*

Proof This follows from the fact that Φ is a (classical) crystal isomorphism. Indeed, the map Ψ is well-defined as a crystal morphism by Lemma 5.4. Let $(\nu, J) \in \mathrm{RC}(\infty)$ and $a \in I$. Next, denote the e_a operator in the crystal X by e_a^X. Then

$$e_a^{\mathcal{T}(\infty)} \Psi(\nu, J) = e_a^{\mathcal{T}(\infty)} \Phi(\nu, J) = e_a^{\mathcal{T}(\lambda_\nu)} \Phi(\nu, J) = e_a^{B^{\otimes \lambda_\nu}} (\nu, J) = e_a^{\mathrm{RC}(\lambda_\nu)} (\nu, J).$$

Since (ν, J) is nonzero in $\mathrm{RC}(\lambda_\nu)$ by definition of λ_ν, we have $e_a^{\mathrm{RC}(\lambda_\nu)}(\nu, J) = e_a^{\mathrm{RC}(\infty)}(\nu, J)$. Thus Ψ commutes with e_a. Showing that Ψ commutes with f_a is similar. □

Remark 5.8 Consider some $T \in B(\infty)$, we can explicitly describe the image of T when projecting under p_λ to $B(\lambda)$ for all $\lambda \in P^+$. From this, we define an equivalence class $[T] = \{p_\lambda(T) : p_\lambda \neq 0,\ \lambda \in P^+\}$. We can check that $[T]$ corresponds to the class of all valid rigged configurations under Φ.

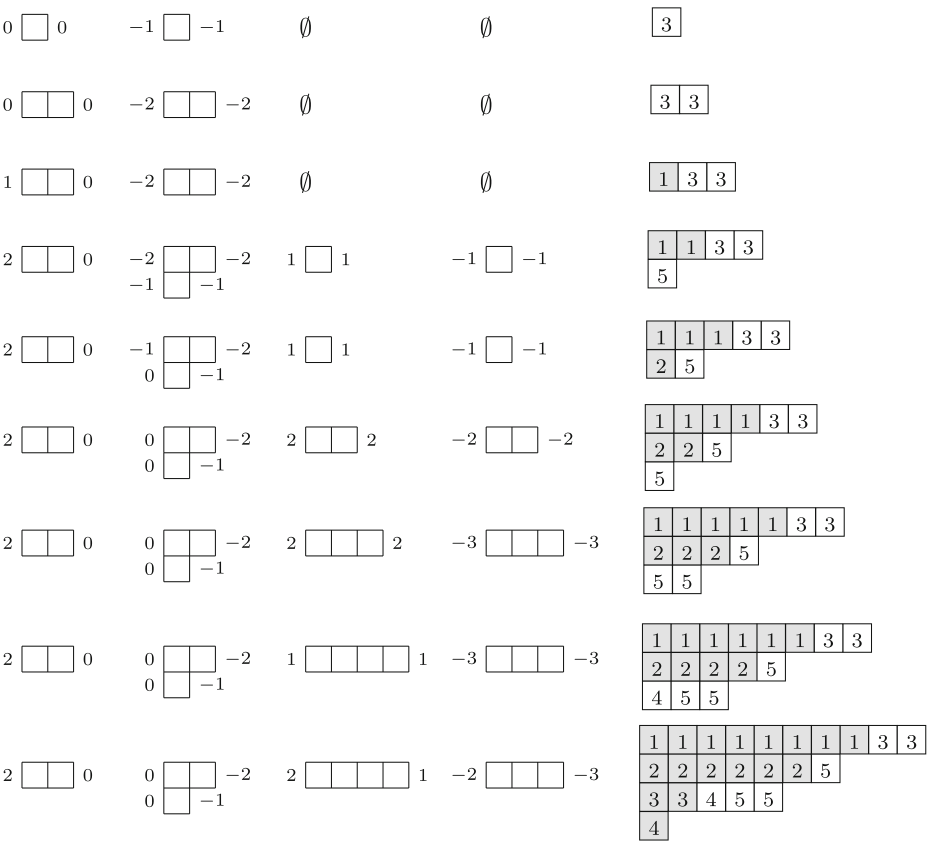
Example 5.9 Let $\mathfrak{g}$ be of type A_4 and

$T =$

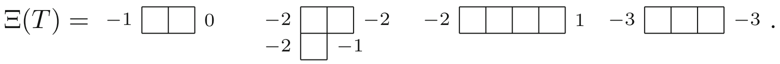
1	1	1	1	1	1	1	1	3	3
2	2	2	2	2	2	5			
3	3	4	5	5					
4									

.

We first project onto $B(\lambda) \subseteq B^{\otimes \lambda}$ with $\lambda = \Lambda_4 + 4\Lambda_3 + 2\Lambda_2 + 3\Lambda_1$ (which results in the same tableaux). Next we apply $\Phi \colon B^{\otimes \lambda} \longrightarrow \mathrm{RC}(B^{\otimes \lambda})$, and have the following:



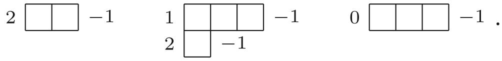
By mapping back the rigged configuration into $\mathrm{RC}(\infty)$, we obtain



In SageMath, we can reproduce the example using

```
sage: B = crystals.infinity.Tableaux("A4")
sage: t = B(rows=[[1,1,1,1,1,1,1,1,3,3],\
....:            [2,2,2,2,2,2,5],[3,3,4,5,5],[4]])
sage: RC = crystals.infinity.RiggedConfigurations("A4")
sage: defn = {B.highest_weight_vector():RC.highest_weight_vector()};
sage: Xi = B.crystal_morphism(defn)
sage: t.pp()
  1  1  1  1  1  1  1  1  3  3
  2  2  2  2  2  2  5
  3  3  4  5  5
  4
sage: ascii_art(Xi(t))
-1[ ][ ]0  -2[ ][ ]-2  -2[ ][ ][ ][ ]1  -3[ ][ ][ ]-3
           -2[ ]-1
```

Example 5.10 Let $\mathfrak{g}$ be of type A_3 and

$(\nu, J) =$ 

Then

$$\Psi(\nu, J) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 2 & 3 & 4 & 4 \\ \cline{1-6} 3 & 4 \\ \cline{1-2} \end{array}.$$

6 Statistics

One of the motivations behind developing a model for $B(\infty)$ in terms of rigged configurations was to describe certain statistics coming from $B(\infty)$ in terms of the combinatorics of rigged configurations. In [1, 2], Brubaker-Bump-Friedberg develop a rule for computing the p-parts of Weyl group multiple Dirichlet series by defining certain decorations on string parametrizations of elements of $B(\lambda)$ due to Kashiwara [19], Berenstein-Zelevinsky [4, 5], and Littelmann [29]. The results of Brubaker-Bump-Friedberg were valid for root systems of type A, and were applied to an expression of the Gindikin-Karpelevich formula as a sum over $B(\infty)$ using these decorated string parameterizations by Bump-Nakasuji [3]. In [30], the decorated string parametrization associated to a vertex in $B(\infty)$ was interpreted in terms of the content of the marginally large tableau at that vertex.

In [31], the statistic in [30] was generalized to cover all remaining types for which Hong-Lee defined their marginally large tableaux in [15]; i.e., types B_n, C_n, D_n, and G_2. In the remainder of this section, we will recall the statistic defined in [30, 31] and explain how it may be interpreted as a statistic on $\mathrm{RC}(\infty)$. The porting of the statistic from marginally large tableaux to rigged configurations is a direct consequence of the crystal isomorphism defined above.

Consider a marginally large tableau $T \in \mathcal{T}(\infty)$. A k-segment in the i-th row of T is a maximal sequence of k-boxes in T such that $i \prec k$, where $\prec$ is the order on the index set of the underlying root system of $\mathcal{T}(\infty)$ given in Eq. 3.1. Let $\mathrm{seg}'(T)$ be the total number of segments of T.

Definition 6.1 ([30, 31]) The segment statistic seg on marginally large tableaux is defined type-by-type as follows.

- A_n: Define $\mathrm{seg}(T) := \mathrm{seg}'(T)$.
- B_n: Let $e_B(T)$ be the number of rows i the contain both a 0-segment and $\bar{\imath}$ segment. Define $\mathrm{seg}(T) := \mathrm{seg}'(T) - e_B(T)$.
- C_n: Define $\mathrm{seg}(T) := \mathrm{seg}'(T)$.
- D_{n+1}: Let $e_D(T)$ be the number of rows i that contain an $\bar{\imath}$-segment but neither a $(n+1)$-segment nor $\overline{n+1}$-segment. Define $\mathrm{seg}(T) = \mathrm{seg}'(T) + e_D(T)$.
- G_2: Let $e_G(T)$ be 1 if T contains a 0-segment and $\bar{1}$-segment in the first row and 0 otherwise. Define $\mathrm{seg}(T) = \mathrm{seg}'(T) - e_G(T)$.

Define $\delta_{(r)}$ by δ on $\mathrm{RC}(\lambda)$ and then embedding into $\mathrm{RC}(\lambda - \Lambda_r)$ where $r = \max\{a \in I : \langle h_a, \lambda \rangle \neq 0\}$. Equivalently $\delta_{(r)} = \delta^r$ but returning only the b from the first application of δ.

Definition 6.2 The *repeat statistic* rpt on rigged configurations is given recursively as follows. Consider $(\nu, J) \in \mathrm{RC}(\lambda_\nu)$. Start with $r = n$ and $s = 0$. Let x_r^c denote the smallest colabel of $\nu^{(r)}$ and project (ν, J) into $\mathrm{RC}(\lambda_{(r)})$, where $\lambda_{(r)} = \lambda - x_r^c \Lambda_r$. Let $c_r = \langle h_r, \lambda_{(r)} \rangle$, and let $b^{(r)} = (b_1, \ldots, b_{c_r})$ be the values returned by $\delta_{(r)}^{c_r}$ (in $\mathrm{RC}(\lambda')$). Increase s by the number of distinct elements occurring in b (note any particular value in b occurs sequentially). We also do the following modifications depending on the type:

B_n: If $0, \overline{r} \in b$, subtract 1 from s.
D_{n+1}: If $\overline{r} \in b$ and $n+1, \overline{n+1} \notin b$, then add 1 to s.
G_2: If $0, \overline{1} \in b$, subtract 1 from s

Now recurse with $r-1$ unless $r=1$, and rpt is the final value of s.

From our definition of rpt, for a fixed r we have $\langle h_a, \lambda \rangle = 0$ for all $a > r$. So the values $p_i^{(a)}$ are equal on $\mathrm{RC}(\infty)$ and $\mathrm{RC}(\lambda)$ for all $a > r$ and $i \in \mathbf{Z}$. So the map $\delta_{(r)}$ starts each time at $\nu^{(r)}$ and is well-defined since all strings in $\nu^{(a)}$ for $a < r$ are non-singular after applying δ. Furthermore by Lemma 5.3, the statistic rpt is well-defined. Thus we have the following.

Proposition 6.3 *Let $(\nu, J) \in \mathrm{RC}(\infty)$. Then*

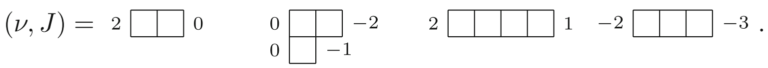
$$\mathrm{rpt}(\nu, J) = \mathrm{seg}(\Psi(\nu, J)).$$

From the definition of rpt, it is clear that it is a direct translation of seg as prescribed by the crystal isomorphism proved above. However, it would be good if there was a non-recursive translation of seg on rigged configurations. In any case, we are afforded the following corollary to this translation of the segment statistic to a statistic on rigged configurations.

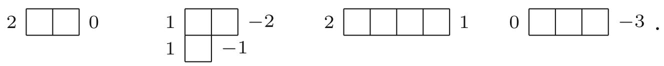
Corollary 6.4 *Let Δ^+ denote the positive roots in a root system of type A_n, B_n, C_n, D_n, or G_2, and let q^{-1} and z be formal parameters. Then*

$$\prod_{\alpha \in \Delta^+} \frac{1 - q^{-1} z^{\alpha}}{1 - z^{\alpha}} = \sum_{(\nu, J) \in \mathrm{RC}(\infty)} (1 - q^{-1})^{\mathrm{rpt}(\nu, J)} z^{-wt(\nu, J)}.$$

Example 6.5 Consider the rigged configuration (ν, J) obtained from Example 5.9:



Thus we begin with $(\nu, J) \in \mathrm{RC}(\lambda_\nu)$ with $\lambda_\nu = 4\Lambda_4 + 5\Lambda_3 + 4\Lambda_2 + 3\Lambda_1$, so we have



We thus project onto $\mathrm{RC}(4\Lambda_3 + 3\Lambda_2 + 2\Lambda_1)$, and since $\langle h_4, \lambda_{(4)} \rangle = 0$, there is nothing more to do. Next, after projecting onto $\mathrm{RC}(3\Lambda_3 + 3\Lambda_2 + 2\Lambda_1)$ and then applying $\delta_{(3)}^3$, we obtain

with $b_{(3)} = (4, 5, 5)$. Projecting onto $\mathrm{RC}(\lambda_2 + 2\Lambda_1)$, we then obtain after applying $\delta_{(2)}$, we obtain

$$0\ \square\square\ 0 \qquad -2\ \square\square\ -2 \qquad \emptyset \qquad \emptyset$$

with $b_{(2)} = (5)$. Finally, we apply $\delta_{(1)}^2$ and obtain $b_{(1)} = (3, 3)$. Therefore, we have $\mathrm{rpt}(\nu, J) = 2 + 1 + 1 = 4$. It is also easy to check that $\mathrm{seg}(T) = 4$.

There is another statistic on rigged configurations which has a natural crystal interpretation. The *difference statistic* diff_a is defined by

$$\mathrm{diff}_a(\nu, J) := \min_i \{p_i^{(a)} - \max J_i^{(a)}\}, \quad (\nu, J) \in \mathrm{RC}(\lambda).$$

The difference statistic measures how far (ν, J) is from being marginally valid in $\nu^{(a)}$, and so it can be interpreted as the largest c_a such that $(\nu, J) \in \mathrm{RC}(\lambda - c_a\Lambda_a)$ (under the natural projection). On the other hand, for general highest weight crystals, define the *remove statistic* $\mathrm{rem}_a(b)$ to be the largest c_a such that $b \in B(\lambda)$ is non-zero under the natural projection to $B(\lambda - c_a\Lambda_a)$. Given these interpretations, the following is an immediate consequence, which arose from the analysis of marginally valid rigged configurations given in the previous sections.

Proposition 6.6 *Let* $\Psi\colon \mathrm{RC}(\lambda) \longrightarrow B(\lambda)$ *be an isomorphism. Then*

$$\mathrm{diff}_a(\nu, J) = \mathrm{rem}_a(\Psi(\nu, J)).$$

Furthermore, the difference statistic can be extended to $\mathrm{RC}(\infty)$, where it measures how far the rigged configuration is from being valid in $\nu^{(a)}$. Thus we define a map $\mathrm{diff}\colon \mathrm{RC}(\infty) \longrightarrow P^+$ by

$$(\nu, J) \mapsto -\sum_{a \in I} \mathrm{diff}_a(\nu, J)\Lambda_a.$$

Note that $\mathrm{diff}(\nu, J)$ denotes the (unique) minimal weight that we need to project $(\nu, J) \in \mathrm{RC}(\infty)$ onto in order to guarantee the result is non-zero. Next we give an extension of rem_a to $B(\infty)$ as follows. We first consider the map $\tau\colon B(\infty) \longrightarrow P^+$ defined by the smallest weight λ such that $b \in B(\infty)$ is non-zero under the natural projection onto $B(\lambda)$. Therefore we can define $\mathrm{rem}_a(b) = \langle h_a, \tau(b) \rangle$. This leads to the following analogue of Proposition 6.6.

Proposition 6.7 *Let* $\Psi\colon \mathrm{RC}(\infty) \longrightarrow B(\infty)$ *be the canonical isomorphism. Then*

$$\mathrm{diff}(\nu, J) = \tau\left(\Psi(\nu, J)\right).$$

In particular, we have

$$\mathrm{diff}_a(\nu, J) = \mathrm{rem}_a\left(\Psi(\nu, J)\right).$$

Note that rem_a may also be interpreted as a statistic on (marginally large) tableaux, with no columns of height n in types B_n or D_{n+1}, as being the number of basic columns of height a (possibly not full height) that can be removed from a tableaux $T \in \mathcal{T}(\lambda)$ and sliding the entries of those rows left such that the result is a classical tableaux. Columns of height

n in type B_n are counted twice, and in type D_n they contribute to both rem_n and rem_{n+1}. Additionally, we can extend this interpretation to $T \in \mathcal{T}(\infty)$.

The difference statistic (and the corresponding map diff) arises naturally during the procedure of the bijection Ψ. The map τ given above is equivalent to the map τ described in [28], and it has also appeared in other natural crystal constructs, such as Kashiwara's $*$-involution [19, 20], the crystal commutor [28], and MV polytopes [17]. This suggests that rigged configurations will have a natural interpretation of the $*$-involution and the crystal commutor. Additionally, there should be a natural combinatorial bijection between rigged configurations and MV polytopes (and perhaps their generalization to KLR polytopes [48]). The authors are currently investigating these connections.

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