



Combinatorics/Lie algebras

The flush statistic on semistandard Young tableaux

*La statistique alignée sur des tableaux de Young semi-standard*Ben Salisbury¹

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ABSTRACT

In this note, a statistic on Young tableaux is defined, which encodes data needed for the Casselman–Shalika formula.

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R É S U M É

Dans cette note est définie une statistique sur les tableaux de Young, encodant les données nécessaires à la formule de Casselman–Shalika.

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Text

In joint work with Kyu-Hwan Lee and Phil Lombardo [14], the author reinterpreted the Casselman–Shalika formula expression due to Brubaker–Bump–Friedberg [3,4] and Bump–Nakasuji [5] as a sum over the crystal graph (based on work of Tokuyama [17]) to a sum over tableaux. The expression given in the work of Bump and Nakasuji involves taking paths in the graph of a highest weight crystal from a given vertex to the highest weight vector and decorating the path. These decorations, called boxing and circling, prescribe contributions at a vertex in the form of Gauss sums (coming from the theory of Weyl group multiple Dirichlet series and Whittaker functions). The resulting function, formed by summing the contributions over the crystal together with their respective weights, has been coined a *Tokuyama function*.

The benefit to a tableaux description of the Tokuyama function means that, in practice, one no longer needs to compute the entire path to the highest weight vertex in the crystal graph, which may be very large. Instead, one can extract the essential data from the content of the tableaux at the vertex and obtain the same function. This tableaux description was explained in [14] by using reparametrizations $\mathbf{a}(T)$ and $\mathbf{b}(T)$ of the string parameterization obtained directly from data in a tableau T , but again required the calculation of a sequence (in our case, two sequences). It is the goal of this work to interpret these two sequences as statistics on tableaux.

Such a statistic was created in the context of the Gindikin–Karpelevich formula, again based on the work of Brubaker–Bump–Friedberg [3,4] and Bump–Nakasuji [5]. This formula, from the context of crystals, may be viewed as the Verma module analogue of the highest weight calculation used in the Casselman–Shalika formula. In [15,16], we were able to recover the path to the highest weight vector using the marginally large tableaux of J. Hong and H. Lee [10], which is a certain enlargement of semistandard Young tableaux, and interpret the decorations. The corresponding statistic was called the *segment* statistic and may be easily read off from marginally large tableaux. Outside of type A_r , the proof in [16] relied

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on the interpretation of the Gindikin–Karpelevich formula as a sum over Lusztig’s canonical basis given by H.H. Kim and K.-H. Lee in [12,13] and did not require any decorated paths to the highest weight in the crystal graph.

From the decorated path point of view, the Gindikin–Karpelevich formula only requires the circling rule, and the segment statistic on marginally large tableaux completely encodes the circling data. Moreover, there exists an embedding from semistandard Young tableaux to marginally large tableaux which preserves the path to the highest weight. More precisely, if $\mathcal{B}(\lambda + \rho)$ is the crystal of highest weight $\lambda + \rho$ parameterized by semistandard Young tableaux of shape $\lambda + \rho$, then there is an embedding into the crystal of marginally large tableaux $\mathcal{B}(\infty)$ such that the circling rule is preserved. Understanding this embedding leads one to a definition of segments on ordinary semistandard Young tableaux of fixed shape, so that one is only left to understand the boxing rule. It is the latter problem where the results from [14] become crucial, as the definition of the sequences defined there lead to a descriptive picture of what the boxing rule means on the tableaux level.

The way to understand the boxing rule again involves the idea of segments in a tableau, but it is how these segments are arranged in the tableau which will encode the boxing rule. In this note, we define the *flush* statistic on semistandard Young tableaux T in $\mathcal{B}(\lambda + \rho)$, which, loosely speaking, is the number of segments in T whose left-most box in the segment is in the same column as the left-most box of the subsequent segment in the row beneath it (in English notation for tableaux). In other words, the number of segments who are flush-left with their neighbor below. The notions of “subsequent segment” and “neighbor below” are made precise in Definition 2.3(2) below. It turns out that $\text{flush}(T)$ is exactly the number of boxed entries of $\mathbf{a}(T)$ and is equal to the number of boxed entries in the decorated path from T to the highest weight vector of $\mathcal{B}(\lambda + \rho)$.

It is the hope that the statistics developed here will help shed some new light on the Casselman–Shalika formula outside of type A_r . Currently, there are boxing and circling rules defined and verified in types B_r [7] and C_r [1,2], but only conjectural formulas in types D_r [6] and type G_2 [8].

1. Crystals and tableaux

We start by recalling the setup from [14]. Let $r \geq 1$ and suppose $\mathfrak{g} = \mathfrak{sl}_{r+1}$ with simple roots $\{\alpha_1, \dots, \alpha_r\}$, and let $I = \{1, \dots, r\}$. Let P and P^+ denote the weight lattice and the set of dominant integral weights, respectively. Denote by Φ and Φ^+ , respectively, the set of roots and the set of positive roots. Let $\{h_1, \dots, h_r\}$ be the set of coroots and define a pairing $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbf{Z}$ by $\langle h, \lambda \rangle = \lambda(h)$, where P^\vee is the dual weight lattice. Let $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ be the Cartan subalgebra, and let \mathfrak{h}^* be its dual. Denote the *Weyl vector* by ρ ; this is the element $\rho \in \mathfrak{h}^*$ defined as $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i=1}^r \omega_i$, where ω_i is the i th fundamental weight. The set of roots for \mathfrak{g} will be denoted by Δ , while Δ^+ will denote the set of positive roots and $N = \#\Delta^+$.

A \mathfrak{g} -crystal is a set \mathcal{B} together with maps $\tilde{e}_i, \tilde{f}_i: \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$, $\varepsilon_i, \varphi_i: \mathcal{B} \rightarrow \mathbf{Z} \sqcup \{-\infty\}$, and $\text{wt}: \mathcal{B} \rightarrow P$, such that, for all $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$, and $\langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$. The maps \tilde{e}_i (resp. \tilde{f}_i) for $i \in I$ are called the *Kashiwara raising operators* (resp. *Kashiwara lowering operators*). (For more details, see, for example, [9,11].) To each highest weight representation $V(\lambda)$ of \mathfrak{g} , there is an associated highest weight crystal $\mathcal{B}(\lambda)$ which serves as a combinatorial frame of the representation $V(\lambda)$. The only fact we will use in this note is that $\mathcal{B}(\lambda)$ as a set may be realized as the set of semistandard tableaux of shape λ over the alphabet $\{1, \dots, r + 1\}$ with the usual ordering, where $\lambda = a_1 \omega_1 + \dots + a_r \omega_r$ is identified with the partition having a_i columns of height i , for each $1 \leq i \leq r$.

2. Using the tableaux model

We now recall the definitions and result from [14].

Definition 2.1. (See [14].) Let $\lambda \in P^+$ and $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

(1) Define $\mathbf{a}_{i,j}$ to be the number of $(j + 1)$ -colored boxes in rows 1 through i for $1 \leq i \leq j \leq r$, and define the vector $\mathbf{a}(T) \in \mathbf{Z}_{\geq 0}^N$ by

$$\mathbf{a}(T) = (\mathbf{a}_{1,1}, \mathbf{a}_{1,2}, \dots, \mathbf{a}_{1,r}; \mathbf{a}_{2,2}, \dots, \mathbf{a}_{2,r}; \dots; \mathbf{a}_{r,r}).$$

(2) The number $\mathbf{b}_{i,j}$ is defined to be the number of boxes in the i th row which have color greater or equal to $j + 1$ for $1 \leq i \leq j \leq r$. Set

$$\mathbf{b}(T) = (\mathbf{b}_{1,1}, \dots, \mathbf{b}_{1,r}; \mathbf{b}_{2,2}, \dots, \mathbf{b}_{2,r}; \dots; \mathbf{b}_{r,r}).$$

(3) Write $\lambda + \rho$ as $\lambda + \rho = (\ell_1 > \ell_2 > \dots > \ell_r > \ell_{r+1} = 0)$, and define $\theta_i = \ell_i - \ell_{i+1}$ for $i = 1, \dots, r$. Let $\theta = (\theta_1, \dots, \theta_r)$.

In [14], we give a definition of boxing and circling on the entries of $\mathbf{a}(T) = (\mathbf{a}_{i,j})$ for $T \in \mathcal{B}(\lambda + \rho)$ based on the boxing and circling decorations on BZL paths in [3,4].

$$\text{Box } \mathbf{a}_{i,j} \quad \text{if } \mathbf{b}_{i,j} \geq \theta_i + \mathbf{b}_{i+1,j+1}. \tag{B-II}$$

$$\text{Circle } \mathbf{a}_{i,j} \quad \text{if } \mathbf{a}_{i,j} = \mathbf{a}_{i-1,j}. \tag{C-II}$$

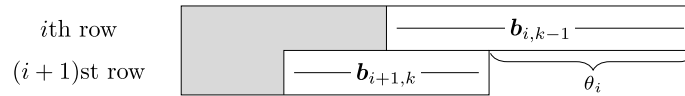


Fig. 2.1. A diagram motivating the definition of flush.

Fig. 2.1. Un diagramme qui motive la définition de alignée.

Set $\text{non}(T)$ to be the number of entries in $\mathbf{a}(T)$ which are neither circled nor boxed, and define $\text{box}(T)$ to be the number of entries in $\mathbf{a}(T)$ which are boxed. Additionally, borrowing the vernacular of the Gelfand–Tsetlin pattern setting of Tokuyama [17], we say that T is *strict* if $\mathbf{a}(T)$ has no entry which is both boxed and circled. Now define a function $C_{\lambda+\rho, q}$ on $\mathcal{B}(\lambda + \rho)$ with values in $\mathbf{Z}[q^{-1}]$ by

$$C_{\lambda+\rho}(T; q) = \begin{cases} (-q^{-1})^{\text{box}(T)}(1 - q^{-1})^{\text{non}(T)} & \text{if } T \text{ is strict,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.2. (See [14].) We have

$$\mathbf{z}^\rho \chi_\lambda(\mathbf{z}) \prod_{\alpha > 0} (1 - q^{-1} \mathbf{z}^{-\alpha}) = \sum_{T \in \mathcal{B}(\lambda+\rho)} C_{\lambda+\rho}(T; q) \mathbf{z}^{\text{wt}(T)}. \tag{2.1}$$

The main result of this note is a new statistic on $T \in \mathcal{B}(\lambda + \rho)$ to compute $C_{\lambda+\rho}(T; q)$ without the need to construct the sequence $\mathbf{a}(T)$.

Definition 2.3. Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

- (1) Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableaux. We define a k -segment [15,16] of T (in the i th row) to be a maximal consecutive sequence of k -boxes in the i th row for any $i + 1 \leq k \leq r + 1$. Denote the total number of k -segments in T by $\text{seg}(T)$.
- (2) Let $1 \leq i < k \leq r + 1$ and suppose ℓ is the smallest integer greater than k such that there exists an ℓ -segment in the $(i + 1)$ st row of T . A k -segment in the i th row of T is called *flush* if the leftmost box in the k -segment and the leftmost box of the ℓ -segment are in the same column of T . If, however, no such ℓ exists, then this k -segment is said to be *flush* if the number of boxes in the k -segment is equal to θ_i . Denote the number of flush k -segments in T by $\text{flush}(T)$.

Example 2.4. Let $r = 4$, $\lambda = 2\omega_3$, and

$$T = \begin{array}{cccccc} \boxed{1} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \boxed{2} & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & \\ \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & & \\ \boxed{5} & & & & & \end{array}.$$

It is easy to see that $\text{seg}(T) = 7$ because there is a 2-segment in the first row, a 3-segment in both the first and second rows, a 4-segment in the second row, and a 5-segment in each of the first, third, and fourth rows. Moreover, $\text{flush}(T) = 5$ because each 3-segment and 5-segment is flush. In other words, the 2-segment in the first row and the 4-segment in the second row are not flush.

The motivating picture for the definition of flush is given in Fig. 2.1, and will be useful to keep in mind for the proof of Theorem 2.5.

Theorem 2.5. Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

- (1) Let $1 \leq i < k \leq r$. Suppose the following two conditions hold.
 - (a) There is no k -segment in the i th row of T .
 - (b) Let ℓ be the smallest integer greater than k such that there exists an ℓ -segment in the i th row. There is no p -segment in the $(i + 1)$ st row, for $k + 1 \leq p \leq \ell$, and the ℓ -segment is flush.²
 Then $C_{\lambda+\rho}(T; q) = 0$.
- (2) If conditions (1a) and (1b) are not satisfied, then

$$C_{\lambda+\rho}(T; q) = (-q^{-1})^{\text{flush}(T)} (1 - q^{-1})^{\text{seg}(T) - \text{flush}(T)}.$$

² By convention, if no such ℓ exists, then condition (1b) is not satisfied.

Proof. First note that $\mathbf{b}_{i,k-1} \geq \mathbf{b}_{i+1,k} + \theta_i$ implies $\mathbf{b}_{i,k-1} = \mathbf{b}_{i+1,k} + \theta_i$ because T is semistandard.

We claim that conditions (1a) and (1b) are equivalent to $\mathbf{a}_{i,k-1}$ in $\mathbf{a}(T)$ being both boxed and circled. First, there is no k -segment in the i th row if and only if $\mathbf{a}_{i,k-1} = \mathbf{a}_{i-1,k-1}$, which justifies condition (1a). It now remains to show that (1b) is equivalent to $\mathbf{a}_{i,k-1}$ being boxed. If condition (1b) holds, then, by the definition of ℓ and being flush, we have $\mathbf{b}_{i,k-1} = \mathbf{b}_{i,\ell-1} = \mathbf{b}_{i+1,\ell} + \theta_i = \mathbf{b}_{i+1,k} + \theta_i$, so $\mathbf{a}_{i,k-1}$ is boxed. On the other hand, if $\mathbf{a}_{i,k-1}$ is boxed and there is no k -segment in the i th row, then $\mathbf{b}_{i,k-1} = \mathbf{b}_{i,\ell-1} = \mathbf{b}_{i+1,k} + \theta_i$, where ℓ is as in condition (1b). The only way $\mathbf{b}_{i,\ell-1} = \mathbf{b}_{i+1,k} + \theta_i$ is if the leftmost box of the ℓ -segment in the i th row and the leftmost box of the m -segment in the $(i + 1)$ st row are in the same column, where m is the smallest integer greater than k such that there exists an m -segment in the $(i + 1)$ st row. By the semistandardness of T , this implies that condition (1b) must be satisfied.

To see condition (2), it follows from Lemma 2.5 and Proposition 2.7 of [14] that $\text{seg}(T)$ is exactly the number of entries in $\mathbf{a}(T)$ which are not circled. Additionally, it follows immediately from the definition that a k -segment in the i th row is flush if and only if $\mathbf{b}_{i,k-1} = \mathbf{b}_{i+1,k} + \theta_i$. Hence $\text{box}(T) = \text{flush}(T)$ and $\text{non}(T) = \text{seg}(T) - \text{flush}(T)$, as required. \square

Definition 2.6. If $T \in \mathcal{B}(\lambda + \rho)$ is a tableau and if conditions (1a) and (1b) in Theorem 2.5 are satisfied for some $1 \leq i < k \leq r$, then we say that T has gaps. If no such pair (i, k) satisfies conditions (1a) and (1b) in Theorem 2.5, then we say that T is gapless.

Using the idea of gaps, we may rewrite

$$C_{\lambda+\rho}(T; q) = \begin{cases} (-q^{-1})^{\text{flush}(T)}(1 - q^{-1})^{\text{seg}(T) - \text{flush}(T)} & \text{if } T \text{ is gapless,} \\ 0 & \text{if } T \text{ has gaps,} \end{cases}$$

to get

$$z^\rho \chi_\lambda(\mathbf{z}) \prod_{\alpha > 0} (1 - q^{-1} z^{-\alpha}) = \sum_{\substack{T \in \mathcal{B}(\lambda+\rho) \\ T \text{ gapless}}} (-q^{-1})^{\text{flush}(T)} (1 - q^{-1})^{\text{seg}(T) - \text{flush}(T)} \mathbf{z}^{\text{wt}(T)}. \tag{2.2}$$

Example 2.7. Let

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 2 & 4 & \\ \hline 3 & 4 & & & \\ \hline \end{array}.$$

There is no 2-segment in the first row nor is there a 3-segment in the second row. However, the 3-segment in the first row is flush, so T has a gap and $C_{\lambda+\rho}(T; q) = 0$. As a check, we have $\mathbf{a}(T) = (0, 1, 1; 1, 2; 3)$ and $\mathbf{b}(T) = (2, 2, 1; 1, 1; 1)$. By (C-II), $\mathbf{a}_{1,1} = 0$ is circled because the (non-existent) entry $\mathbf{a}_{0,1} = 0$. Moreover, by (B-II), $\mathbf{a}_{1,1}$ is boxed because $\mathbf{b}_{1,1} = 2$ and $\mathbf{b}_{2,2} + \theta_1 = 1 + 1$.

Example 2.8. Let

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ \hline 2 & 2 & 3 & 3 & 3 & 4 & & \\ \hline 4 & 4 & 4 & & & & & \\ \hline \end{array}.$$

All possible k -segments are included in T , so condition (1a) of Theorem 2.5 is not satisfied, so T is gapless and $C_{\lambda+\rho}(T; q) \neq 0$. There is a 2-segment in the first row, a 3-segment in both the first and second rows, and a 4-segment in all three rows. Thus $\text{seg}(T) = 6$. Next, the 2-segment in the first row, the 3-segment in the first row, and 4-segment in the last row are each flush, so $\text{flush}(T) = 3$. Hence $C_{\lambda+\rho}(T; q) = (-q^{-1})^3(1 - q^{-1})^3$. As a check, $\mathbf{a}(T) = (3, 2, 1; 5, 2; 5)$, where no entry is circled and $\mathbf{a}_{1,1}$, $\mathbf{a}_{1,2}$, and $\mathbf{a}_{3,3}$ are all boxed, as required.

Remark 2.9. While the proof above made use of the circling and boxing rules of [4], the statistics $\text{seg}(T)$ and $\text{flush}(T)$ are intrinsic to the tableaux. Thus, generalizing these statistics to other Lie algebras may yield an appropriate Casselman–Shalika formula expansion over a crystal graph without the need for circling and boxing rules. At this moment, such an expansion is important as there are no proven circling and boxing rules in types D_r and G_2 . (See [6] for more on the conjecture in type D_r and [8] for the conjecture in type G_2 .)

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References

- [1] J. Beineke, B. Brubaker, S. Frechette, Weyl group multiple Dirichlet series of type C , *Pac. J. Math.* 254 (1) (2011) 11–46.
- [2] J. Beineke, B. Brubaker, S. Frechette, A crystal definition for symplectic multiple Dirichlet series, in: *Multiple Dirichlet Series, L-Functions and Automorphic Forms*, in: *Prog. Math.*, vol. 300, Birkhäuser/Springer, New York, 2012, pp. 37–63.
- [3] B. Brubaker, D. Bump, S. Friedberg, Weyl group multiple Dirichlet series, Eisenstein series and crystal bases, *Ann. Math. (2)* 173 (1) (2011) 1081–1120.
- [4] B. Brubaker, D. Bump, S. Friedberg, *Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory*, *Ann. Math. Stud.*, vol. AM-175, Princeton Univ. Press, New Jersey, 2011.
- [5] D. Bump, M. Nakasuji, Integration on p -adic groups and crystal bases, *Proc. Am. Math. Soc.* 138 (5) (2010) 1595–1605.
- [6] G. Chinta, P.E. Gunnells, Littelmann patterns and Weyl group multiple Dirichlet series of type D , in: *Multiple Dirichlet Series, L-Functions and Automorphic Forms*, in: *Prog. Math.*, vol. 300, Birkhäuser/Springer, New York, 2012, pp. 119–130.
- [7] S. Friedberg, L. Zhang, Eisenstein series on covers of odd orthogonal groups, arXiv:1301.3026.
- [8] H. Friedlander, L. Gaudet, P.E. Gunnells, Crystal graphs, Tokuyama’s theorem, and the Gindikin–Karpelevič formula for G_2 , arXiv:1402.0411.
- [9] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, *Grad. Stud. Math.*, vol. 42, American Mathematical Society, Providence, RI, 2002.
- [10] J. Hong, H. Lee, Young tableaux and crystal $\mathcal{B}(\infty)$ for finite simple Lie algebras, *J. Algebra* 320 (2008) 3680–3693.
- [11] M. Kashiwara, On crystal bases, in: *Representations of Groups*, Banff, AB, 1994, in: *CMS Conf. Proc.*, vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.
- [12] H.H. Kim, K.-H. Lee, Representation theory of p -adic groups and canonical bases, *Adv. Math.* 227 (2) (2011) 945–961.
- [13] H.H. Kim, K.-H. Lee, Quantum affine algebras, canonical bases, and q -deformation of arithmetical functions, *Pac. J. Math.* 255 (2) (2012) 393–415.
- [14] K.-H. Lee, P. Lombardo, B. Salisbury, Combinatorics of the Casselman–Shalika formula in type A , *Proc. Am. Math. Soc.* (2014), <http://dx.doi.org/10.1090/S0002-9939-2014-11961-7>, in press, arXiv:1111.1134.
- [15] K.-H. Lee, B. Salisbury, A combinatorial description of the Gindikin–Karpelevich formula in type A , *J. Comb. Theory, Ser. A* 119 (2012) 1081–1094.
- [16] K.-H. Lee, B. Salisbury, Young tableaux, canonical bases, and the Gindikin–Karpelevich formula, *J. Korean Math. Soc.* 51 (2) (2014) 289–309.
- [17] T. Tokuyama, A generating function of strict Gelfand patterns and some formulas on characters of general linear groups, *J. Math. Soc. Jpn.* 40 (4) (1988) 671–685.